Прилози, *Одд. маш. шех. науки*, МАНУ, **XXX**, 1–2 (2009), стр. 35–52
Contributions, *Sec. Math. Tech. Sci.*, MANU, **XXX**, 1–2 (2009), pp. 35–52
ISSN 0351–3246
UDC: 512.532.2

Original scientific paper

FREE $(n, n+k)$-SEMIGROUPS

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A b s t r a c t: In this paper we introduce the notion of $(n, n+k)$-semigroups, prove some properties about them, and give an algorithmic description of a free $(n, n+k)$-semigroup with a given basis.

Key words: $(n, n+k)$-semigroups; free $(n, n+k)$-semigroups

1. $(n, n+k)$-SEMIGROUPS

Definition 1. Let $n, k \in \mathbb{N}$, and let $G \neq \varnothing$. If $f : G^n \to G^{n+k}$, then we say that f is na $(n, n+k)$-operation, and that the pair (G, f) is an $(n, n+k)$-groupoid. An $(n, n+k)$-groupoid is called $(n, n+k)$-semigroup, if for each integer $0 \leq p \leq k$,

$$(1^p \times f \times 1^{k-p}) \circ f = (1^k \times f) \circ f,$$

where $1^p \times f \times 1^{k-p} : G^{n+k} \to G^{n+2k}$ is defined by:

$$1^p \times f \times 1^{k-p}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{u}, f(\boldsymbol{v}), \boldsymbol{w}),$$

for each $\boldsymbol{u} \in G^p$, $\boldsymbol{v} \in G^n$, and $\boldsymbol{w} \in G^{k-p}$.

Example 1. Let $n = 1$, $k = 1$, $G = \{a, b, c\}$, and let $f : G^1 \to G^2$ be defined by: $f(a) = (b, c)$, $f(b) = (b, d)$, $f(c) = (d, c)$ and $f(d) = (d, d)$. From the definition it follows that:

$(f\times 1^1)\circ f(a) = (f\times 1^1)(b, c) = (b, d, c) = (1^1\times f)(b, c) = (1^1\times f)\circ f(a)$

$(f\times 1^1)\circ f(b) = (f\times 1^1)(b, d) = (b, d, d) = (1^1\times f)(b, d) = (1^1\times f)\circ f(b)$

$(f\times 1^1)\circ f(c) = (f\times 1^1)(d, c) = (d, d, c) = (1^1\times f)(d, c) = (1^1\times f)\circ f(c)$

$(f\times 1^1)\circ f(d) = (f\times 1^1)(d, d) = (d, d, d) = (1^1\times f)(d, d) = (1^1\times f)\circ f(d)$.

This shows that $(f\times 1^1)\circ f = (1^1\times f)\circ f$, i.e. that (G, f) is a (1, 2)-semigroup.

From now on, let (G, f) be an $(n, n + k)$-semigroup.

We define $f^1 = f$, and $f^2 = (1^k\times f)\circ f : G^n \to G^{n+2k}$. The condition that (G, f) is an $(n, n + k)$-semigroup can be stated as: $(1^p\times f\times 1^{k-p})\circ f = f^2$, for each $0 \le p \le k$.

Proposition 1. For any two integers p, q, $0 \le p \le k$ and $0 \le q \le 2k$,

$$(1^p\times f^2\times 1^{k-p})\circ f = (1^q\times f\times 1^{2k-q})\circ f^2.$$

Proof. (*a*) $(1^p\times f^2\times 1^{k-p})\circ f = (1^p\times ((1^{k-p}\times f\times 1^p)\circ f)\times 1^{k-p})\circ f$

$= (1^p\times 1^{k-p}\times f\times 1^p\times 1^{k-p})\circ (1^p\times f\times 1^{k-p})\circ f$

$= (1^k\times f\times 1^k)\circ ((1^p\times f\times 1^{k-p})\circ f)$

$= (1^k\times f\times 1^k)\circ f^2 = (1^k\times f\times 1^k)\circ ((1^k\times f)\circ f) = ((1^k\times f\times 1^k)\circ (1^k\times f))\circ f$

$= (1^k\times ((f\times 1^k)\circ f))\circ f = (1^k\times f^2)\circ f$.

(b) In (a) we have proved that

$$(1^k\times f\times 1^k)\circ f^2 = (1^k\times f^2)\circ f = (1^p\times f^2\times 1^{k-p})\circ f,$$

i.e. we have proved the Proposition for $q = k$.

(c) Next, let $q \ne k$.

If $q < k$, then $2k - q = k + (k - q)$, $0 < k - q < k$, and

$(1^q\times f\times 1^{2k-q})\circ f^2 = (1^q\times f\times 1^{2k-q})\circ (1^q\times f\times 1^{k-q})\circ f$

$= (1^q\times f\times 1^k\times 1^{k-q})\circ (1^q\times f\times 1^{k-q})\circ f$

$= (1^q \times ((f \times 1^k) \circ f) \times 1^{k-q}) \circ f = (1^q \times ((1^k \times f) \circ f) \times 1^{k-q}) \circ f$

$= (1^q \times f^2 \times 1^{k-q}) \circ f = (1^k \times f^2) \circ f = (1^p \times f^2 \times 1^{k-p}) \circ f.$

If $q > k$, then $q = k + (q - k)$, $2k - q < k$, $0 < q - k < k$, and

$(1^q \times f \times 1^{2k-q}) \circ f^2 = (1^q \times f \times 1^{2k-q}) \circ (1^{q-k} \times f \times 1^{2k-q}) \circ f$

$= (1^{q-k} \times 1^k \times f \times 1^{2q-k}) \circ (1^{q-k} \times f \times 1^{2k-q}) \circ f$

$= (1^{q-k} \times ((1^k \times f) \circ f) \times 1^{2k-q}) \circ f = (1^{q-k} \times f^2 \times 1^{2k-q}) \circ f$

$= (1^k \times f^2) \circ f = (1^p \times f^2 \times 1^{k-p}) \circ f.$ □

We define $f^3 = (1^k \times f^2) \circ f : G^n \to G^{n+3k}$. Then Proposition 1 can be restated as:

Proposition 1'. For any two integers p, q, $0 \le p \le k$ and $0 \le q \le 2k$,

$$(1^p \times f^2 \times 1^{k-p}) \circ f = f^3 = (1^q \times f \times 1^{2k-q}) \circ f^2. \ \square$$

Next we continue by induction. Let $f^{t-1}: G^n \to G^{n+(t-1)k}$ be defined, and let for any two integers p, q, $0 \le p \le k$ and $0 \le q \le (t-1)k$,

$$(1^p \times f^{t-1} \times 1^{k-p}) \circ f = (1^q \times f \times 1^{(t-1)k-q}) \circ f^{t-1}.$$

We define $f^t = (1^k \times f^{t-1}) \circ f : G^n \to G^{n+tk}$.

Proposition 2. For any two integers p, q, $0 \le p \le k$ and $0 \le q \le tk$,

$$(1^p \times f^t \times 1^{k-p}) \circ f = (1^q \times f \times 1^{tk-q}) \circ f^t.$$

Proof. (a) $(1^p \times f^t \times 1^{k-p}) \circ f = (1^p \times (1^{k-p} \times f^{t-1} \times 1^p) \circ f) \times 1^{k-p}) \circ f$

$= (1^p \times 1^{k-p} \times f^{t-1} \times 1^p \times 1^{k-p}) \circ (1^p \times f \times 1^{k-p}) \circ f$

$= (1^k \times f^{t-1} \times 1^k) \circ (1^p \times f \times 1^{k-p}) \circ f = (1^k \times f^{t-1} \times 1^k) \circ f^2$

$= (1^k \times f^{t-1} \times 1^k) \circ ((1^k \times f) \circ f) = ((1^k \times f^{t-1} \times 1^k) \circ (1^k \times f)) \circ f$

$= (1^k \times ((f^{t-1} \times 1^k) \circ f)) \circ f = (1^k \times ((1^k \times f^{t-1}) \circ f)) \circ f = (1^k \times f^t) \circ f.$

(b) Next, let $q = sk + r$, where $s < t$ and $0 \leq r \leq k$. Then:

$$(1^q \times f \times 1^{tk-q}) \circ f^t = (1^r \times 1^{sk} \times f \times 1^{(t-s-1)k} \times 1^{k-r}) \circ f^t$$
$$= (1^r \times (1^{sk} \times f \times 1^{(t-s-1)k}) \times 1^{k-r}) \circ ((1^r \times f^{t-1} \times 1^{k-r}) \circ f)$$
$$= ((1^r \times (1^{sk} \times f \times 1^{(t-s-1)k}) \times 1^{k-r}) \circ (1^r \times f^{t-1} \times 1^{k-r})) \circ f$$
$$= (1^r \times ((1^{sk} \times f \times 1^{(t-s-1)k}) \circ f^{t-1}) \times 1^{k-r}) \circ f$$
$$= (1^r \times f^t \times 1^{k-r}) \circ f = (1^k \times f^t) \circ f$$

The aim of this paper is to give a description of a free $(n, n+k)$-semigroup with a given basis A.

Example 2. The (1, 2)-semigroup (G, f) *in* Example 1, is a free (1, 2)-semigroup with a basis $\{a\}$. Let as show this. Let (H, h) be a (1, 2)-semigroup, $\psi:\{a\} \to H$ be a given map, $a' = \psi(a)$, $h(a') = (b', c')$, $h(b') = (x', y')$, and $h(c') = (u', v')$. Since (H, h) is a (1, 2)-semigroup, it follows that $x' = b'$, $y' = u'$ and $v' = c'$. We extend ψ to the map $\varphi : G \to H$ defined by $\varphi(b) = b'$, $\varphi(c) = c'$ and $\varphi(d) = d'$. This extension is a (1, 2)-*homomorphism*, and is unique with this property.

In the next example we will explain the main idea for the rest of the paper.

Example 3. Let (G, f) be a (2, 3)-semigroup, $\mathbf{x} \in G^2$ and $f(\mathbf{x}) = (a, b, c)$. Then, using Propositons 1 and 2, it follows that:

$f((a, b)) = (a, u, v)$, $f((b, c)) = (u, v, c)$, $f((a, u)) = (a, u, w)$,

$f((v, c)) = (w, v, c)$, $f((u, v)) = (u, w, v)$, $f((a, b)) = (a, u, v)$,

$f((u, w)) = (u, w, w)$, $f((w, v)) = (w, w, v)$, and $f((w, w)) = (w, w, w)$.

This shows that for a given $\mathbf{x} \in G^2$, after several steps no new elements appear in the images $f^t(\mathbf{x})$. In this example, in the images $f^t(\mathbf{x})$ there are at most 6 elements. In the following paragraph, we will construct elements that will appear in the images $f^t(\mathbf{x})$.

2. CONSTRUCTION 1

Let $n, k, s \in \mathbb{N}$ such that $(s-2)\,k < n \le (s-1)\,k < n + k \le sk$.

Definition 2. Let x be an element from a set A. We define two sets: $D(x) \subseteq A \times \mathbb{N} \times \mathbb{N}$ and $E(x) \subseteq \{0\} \times A \times \mathbb{N} \times \mathbb{N}$ by:

$D(x) = \{(x, j, i) \mid 1 \le j \le s,\ 1 \le i \le n + 2k - jk\} = D$,

$E(x) = \{(0, x, j, i) \mid 1 \le j,\ 1 \le i \le n + jk\} = E$.

Definition 3. We define a map φ: $E(x) \to D(x)$ as follows:

For $1 \le t \le s$:

$$\varphi\big((0,x,t,i)\big) = \begin{cases} (x, j, i - jk + k) & \text{for } jk - k < i \le jk, 1 \le j < t \\ (x, t, i - tk + k) & \text{for } tk - k < i \le n + k \\ (x, t - j, i - tk + k) & \text{for } n + jk < i \le n + jk + k, 1 \le j < t \end{cases}$$

For $1 \le r$:

$$\varphi\big((0,x,s+r,i)\big) = \begin{cases} \varphi\big((0,x,s,i)\big) & \text{for } 1 \le i \le sk \\ \varphi\big((0,x,s,i - jk)\big) & \text{for } sk + jk - k < i \le sk + jk, 1 \le j \le r \\ \varphi\big((0,x,s,i - rk)\big) & \text{for } sk + rk < i \le n + sk + rk. \end{cases}$$

Proposition 3. For any $n < i \le sk - k$,

$\varphi((0, x, s, i)) = \varphi((0, x, s, i + k))$.

Proof. Since $n < i \le sk - k$, it follows that $n + k < i + k \le sk - k + k = sk \le n + 2k - 1 < n + 2k$. So, the definition of φ for $n + jk < i + k \le n + jk + k$, $j = 1$, implies that:

$\varphi((0, x, s, i + k)) = (x, s - 1, i + k - sk + k) = (x, s - 1, i - sk + 2k)$.

On the other hand, $n < i \le sk - k$, implies that $sk - 2k < n < i \le sk - k$, i.e. $jk - k < i \le jk$ for $j = s - 1$. So, the definition of φ for s, implies that:

$\varphi((0, x, s, i)) = (x, s - 1, i - (s - 1)\,k + k) = (x, s - 1, i - sk + 2k)$. □

Proposition 4.

(1) $\varphi((0, x, t + 1, i)) = \varphi((0, x, t, i))$, for any $1 \leq i < tk$, and

(2) $\varphi((0, x, t + 1, i + k)) = \varphi((0, x, t, i))$, for any $n < i \leq n + tk$.

Proof. (1) For $t \leq s - 1$, since $i \leq tk < (t + 1)k$, from the definiton we have that:

$\varphi((0, x, t + 1, i)) = (x, j, i - jk + k), jk - k < i \leq jk, 1 \leq j \leq t < t + 1;$

$\varphi((0, x, t, i)) = (x, j, i - jk + k)), jk - k < i \leq jk, 1 \leq j < t.$

For $tk - k < i \leq tk$, since $t \leq s - 1$ it follows that $tk - k < i \leq (s - 1) k \leq n + k$, and so:

$\varphi((0, x, t, i)) = (x, t, i - tk + k)) = \varphi((0, x, t + 1, i)).$

For $t = s$, since $i \leq tk = sk$, from the definiton we have that:

$\varphi((0, x, s + 1, i)) = \varphi((0, x, s, i)).$

For $t > s$, $t = s + r$, for some $r > 0$. Since $i \leq sk + rk$ from the definition we have:

$\varphi((0, x, t + 1, i)) = \varphi((0, x, s, i))$, when $i \leq sk$, and

$\varphi((0, x, t + 1, i)) = \varphi((0, x, s, i - jk))$, when $sk + jk - k < i \leq sk + jk$, $1 \leq j \leq r$.

Similarly,

$\varphi((0, x, t, i)) = \varphi((0, x, s, i))$, when $i \leq sk$, and

$\varphi((0, x, t, i)) = \varphi((0, x, s, i - jk))$, when $sk + jk - k < i \leq sk + jk$, $1 \leq j \leq r$.

Hence, for any $1 \leq i < tk$, $\varphi((0, x, t + 1, i)) = \varphi((0, x, t, i))$.

(2) For $t \leq s - 1$, since $n + k < i + k$, from the definition we have that:

$\varphi((0, x, t + 1, i + k)) = (x, t + 1 - j, i + k - tk - k + k),$

for $n + jk < i + k \leq n + jk + k$, and $1 \leq j \leq t < t + 1$; and

$\varphi((0, x, t, i)) = (x, t - (j - 1), i - tk + k),$

for $n + (j - 1) k < i \leq n + (j - 1) k + k$, and $1 \leq j - 1 < t$.

For $n < i \leq n + k$, since $tk - k \leq sk - k - k < n$ from the definition we have that

$\varphi((0, x, t, i)) = (x, t, i - tk + k) = (x, t - (j - 1), i - tk + k).$

For $t = s$,

$$\varphi\big((0,x,t+1,i+k)\big)=\begin{cases}\varphi\big((0,x,s,i+k)\big) & \text{for } n+k<i+k\le sk\\ \varphi\big((0,x,s,i+k-k)\big) & \text{for } sk+k<i+k\le n+sk+k\\ \varphi\big((0,x,s,i+k-k)\big) & \text{for } sk<i+k\le sk+k.\end{cases}$$

Since for $n + k < i + k \le sk$, $n < i \le sk - k$, Proposition 3 implies that

$\varphi((0, x, s, i + k)) = \varphi((0, x, s, i))$.

For $t > s$, $t = s + r$ for some $r > 0$. Since $n + k < i + k \le sk + rk$ from the definition we have:

$$\varphi\big((0,x,t+1,i+k)\big)=\begin{cases}\varphi\big((0,x,s,i+k)\big) & \text{for } n+k<i+k\le sk\\ \varphi\big((0,x,s,i-rk)\big) & \text{for } sk+rk<i\le n+sk+rk\\ \varphi\big((0,x,s,i+k-jk)\big) & \text{for } sk+jk-k<i+k\le sk+jk\\ & \text{and } 1\le j\le r+1;\end{cases}$$

$$\varphi\big((0,x,t,i)\big)=\begin{cases}\varphi\big((0,x,s,i)\big) & \text{for } n<i\le (s-1)k\\ \varphi\big((0,x,s,i-rk)\big) & \text{for } sk+rk<i\le n+sk+rk\\ \varphi\big((0,x,s,i-(j-1)\,k)\big) & \text{for } sk+jk-2k<i\le sk+jk-k\\ & \text{and } 2\le j\le r+1\end{cases}$$

and

$\varphi((0, x, t, i)) = \varphi((0, x, s, i))$, for $sk - k < i \le sk$.

Again, for $n + k < i + k \le sk$, $n < i \le sk - k$, Proposition 3 implies that

$\varphi((0, x, t + 1, i + k)) = \varphi((0, x, s, i + k)) = \varphi((0, x, s, i))$.

Hence, for any $n < i \le n + tk$, $\varphi((0, x, t + 1, i + k)) = \varphi((0, x, t, i))$. □

Often, an element $U = (a_1, a_2, ..., a_p, b_1, b_2, ..., b_q) \in A^{p+q}$ will be denoted by $U = VW$, where $V = (a_1, a_2, ..., a_p)$ and $W = (b_1, b_2, ..., b_q)$, and to indicate that $W \in A^i$, we write $|W| = i$.

Definition 4. We will use the following notations.

(1) For each $1 \le t \le s - 2$

$X_t = ((x, t, 1), (x, t, 2), ..., (x, t, k)) \in D^k$;

$Y_t = ((x, t, n + k - tk + 1), (x, t, n + k - tk + 2), ..., (x, t, n + 2k - tk)) \in D^k$;

$Z_t = ((x, t, k + 1), (x, t, k + 2), ..., (x, t, k + n - tk)) \in D^{n - tk}$.

(2) For each $t = s - 1$

$X_{s-1}= ((x, s - 1, 1), (x, s - 1, 2), ..., (x, s - 1, n - sk + 2k)) \in D^{n - sk + 2k}$;

$Y_{s-1}= ((x,s - 1,k + 1), (x,s - 1,k + 2), ..., (x,s - 1, n +2\text{k}- sk + k)) \in D^{n - sk + 2k}$.

(3) $X = ((x, s - 1, n - sk + 2k + 1), ..., (x, s - 1, k)) \in D^{sk - k - n}$.

$Y = ((x, s, 1), (x, s, 2), ..., (x, s, n + 2k - sk)) \in D^{n + 2k - sk}$.

(4) $X_s = X_1X_2...X_{s-2}X_{s-1}$ and $Y_s = Y_{s-1}Y_{s-2}...Y_2Y_1$.

(5) For each $r \geq 0$, $S_r = XYXY...XY \in D^{rk}$ and $T_r = YXYX...YX \in D^{rk}$.

(6) For each $1 \leq t$,

$M_t = (\varphi((0, x, t, 1)), \varphi((0, x, t, 2)), ...\varphi((0, x, t, n + tk))) \in D^{n + tk}$.

In Definition 4: X_t, Y_t, Z_t are well defined since for $1 \leq t \leq s - 2$,

$(k + n - \text{tk}) - k = n - tk \geq n - (s - 2)k = n + 2k - sk \geq 1$.

Similarly X_{s-1}, Y_{s-1} are well defined since $n - sk + 2k < k + 1$.

For $n < (s - 1)k$, $0 < sk - k - n$, and so $|X| > 0$.

For $n = (s - 1)k$, $0 = sk - k - n$, and so $|X| = 0$, but then $|X_{s-1}| = |Y_{s-1}| = k$.

From Definition 4 it follows that $|X_s| = |Y_s| = n$, $|X_sX| = |XY_s| = sk - k$, and $|XY| = k$.

All the elements in the k – tuples X_t, Y_t, Z_t are distinct, and there are exactly $n + 2k - tk$ of them.

It follows directly from Definition 4, that each element of D, appears exactly once in exactly one of X_t, Y_t, Z_t, X_{s-1}, Y_{s-1}, X, Y.

With the above notations, $S_rX = YT_r$ and $|S_0| = |T_0| = 0$.

Proposition 5.

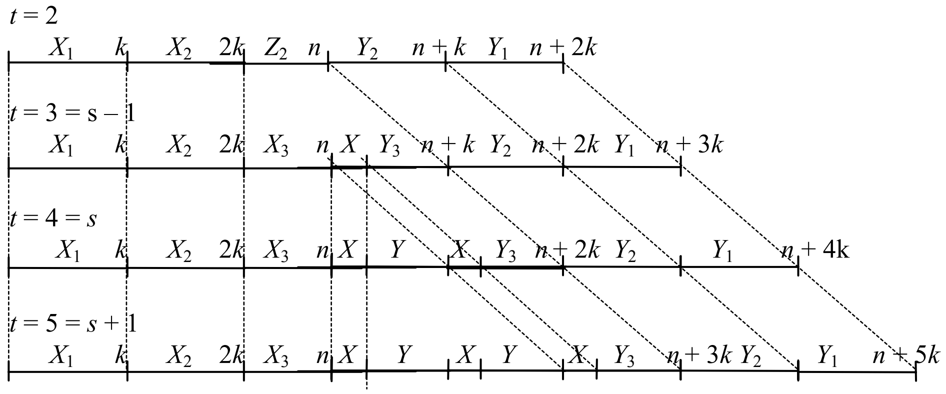
(a) For $1 \leq t \leq s - 2$, $M_t = X_1X_2...X_{t-1}X_tZ_tY_tY_{t-1}...Y_2Y_1$.

(b) $M_{s-1} = X_1X_2...X_{s-2}X_{s-1}XY_{s-1}Y_{s-2}...Y_2Y_1 = X_sXY_s$.

(c) $M_s = X_1X_2...X_{s-2}X_{s-1}XYXY_{s-1}Y_{s-2}...Y_2Y_1 = X_sXYXY_s$.

(d) For $1 \le r$, $M_{s+r} = X_s XYXY...XYXY_s = X_s S_{r+1} XY_s = X_s YT_{r+1} XY_s$. □

Schematically, some of the M_t 's, for $s = 4$, are shown bellow:



Proof. (**a**) and (**b**) follow directly from the definitions, while (**d**) follows from (**c**), the definitions and Proposition 4. For (**c**), the definitions and Proposition 3 imply that:

$(\varphi\,((0, x, s, n+1)), \varphi\,((0, x, s, n+2)), ..., \varphi\,((0, x, s, sk-k)))$

$= ((x, s-1, n-sk+2k+1), ..., (x, s-1, sk-k-sk+2k)) = X.$

For $sk - k < i \le n + k$, the definition of φ implies that:

$(\varphi\,((0, x, s, sk-k+1)), \varphi\,((0, x, s, sk-k+1)), ..., \varphi\,((0, x, s, n+k)) = Y.$ □

In the above proposition, (**b**), (**c**) and (**d**) can be restated as:

for $0 \le r$, $M_{s-1+r} = X_s S_r XY_s = X_s XT_r Y_s$.

Proposition 6. Let $M_t = ULV$, $M_q = PLQ$, where $L \in D^n$ and $t \le q$. We consider the following two cases: $t \le s-2$ *and* $t \ge s-1$.

(a) $t \le s-2$.

In this case, $q = t$, $P = U$, and $Q = V$.

(b) $t \ge s-1$.

In this case we have the following four possibilities.

(b.1) $M_t = UL'L_1L''V$, such that $UL' = X_s$, $L''V = Y_s$, $|L'| > 0$ and $|L''| > 0$. Then $q = t$, $P = U$, and $Q = V$. In this case, $tk < 2n$.

(b.2) $M_t = UL'L_1V'Y_s$, such that $UL' = X_s$, $V = V'Y_s$ and $|L'| > 0$. Then $P = U$, $Q = V'T_{q-t}Y_s$ and $M_q = ULV'T_{q-t}Y_s$.

(b.3) $M_t = X_sU'L_1L''V$, such that $L''V = Y_s$, $U = X_s U'$ and $|L''| > 0$. Then $Q = V$, $P = X_s S_{q-t} U'$ and $M_q = X_s S_{q-t} U'LV$.

(b.4) $M_t = X_sU'LV'Y_s$, such that $U = X_sU'$, $V = V'Y_s$. Then $P = X_sP'$, $Q = Q'Y_s$, $U' = S_rW_1$, $P' = S_pW_1$, $V' = W_2T_i$, $Q' = W_2T_j$, $|W_1| < k$, $|W_2| < k$, $M_t = X_sS_rW_1LW_2T_iY_s$, and $M_q = X_sS_qW_1LW_2T_jY_s$ for some p, q, i, j, W_1 and W_2. In this case, $tk \geq 2n$.

Proof. (a) $M_t = X_1...X_tZ_tY_t...Y_1$ and $|X_1...X_t| = |Y_t...Y_1| = t\,k \leq (s-2)\,k < n$. This implies that L has a part of Z_t. Since the elements of Z_t appear only in M_t it follows that $q = t$. Since the first element of L appears only once in M_t, it follows that $|U| = |P|$. Hence $P = U$, and so $Q = V$.

(b) $M_t = X_sXT_{t+1-s}Y_s = ULV$, $|XT_{t+1-s}| = sk - k - n + (t+1-s)\,k = tk - n \geq sk - k - n \geq 0$. In this case we have the following four subcases:

(b.1) L has parts of both X_s, Y_s, i.e. $L = L'L_1L''$, $XT_{t+1-s} = L_1$, $|L'| > 0$, $|L''| > 0$;

(b.2) L has a part only of X_s, i.e. $L = L'L_1$, $XT_{t+1-s} = L_1V'$, $V = V'Y_s$, $|L'| > 0$;

(b.3) L has a part only of Y_s, i.e. $L = L_1L''$, $S_{t+1-s}X = U'L_1$, $U = X_s U'$, $|L''| > 0$;

(b.4) L has no parts of X_s, Y_s, i.e. $U = X_s U'$, $V = V'Y_s$, $|U'| \geq 0$, $|V'| \geq 0$.

(b.1) In this case: $|L| = n > |XT_{t+1-s}| = tk - n$, i.e. $tk < 2n$; $X_s = UL'$; and $Y_s = L''V$. The first element of L' is in X_s , the last element of L'' is in Y_s and they appear only once. Hence, $P = U$, $Q = V$, and $M_q = ULV = M_t$, i.e. $q = t$.

(b.2) In this case: $X_s = UL'$ and $M_t = UL'L_1V'Y_s$. The first element of L' is in X_s and appears only once. So, $P = U$ and $M_q = UL'L_1Q = X_sL_1WY_s$ where $Q = WY_s$ and $L_1W = XT_{q+1-s} = XT_{t+1-s}\,T_{q-t} = L_1V'T_{q-t}$. This implies that $W = V'T_{q-t}$ and $Q = V'T_{q-t}\,Y_s$.

(b.3) In this case: $Y_s = L''V$; and $M_t = X_sU'L_1L''V$. The last element of L'' is in Y_s and appears only once. So $Q = V$ and $M_q = PL_1L''V = X_sWL_1Y_s$ where $P = X_sW$ and $WL_1 = S_{q+1-s}X = S_{q-t}\,S_{t+1-s}X = S_{q-t}\,U'L_1$. This implies that $W = S_{q-t}\,U'$, $P = X_s\,S_{q-t}U'$ and $M_q = X_s\,S_{q-t}\,U'LV$.

(b.4) In this case: $S_{t+1-s}X = XT_{t+1-s} = U'LV'$. Since L has no parts of X_s and Y_s, it follows that $S_{q+1-s}X = XT_{q+1-s} = P'LQ'$, $P = X_sP'$, $Q = Q'Y_s$ and $M_q = X_s P'LQ'Y_s$. Let: $U' = S_r U''$ and $P' = S_p P''$ where $|P''| < k$ and $|U''| < k$. Then, $S_{t+1-s} X = U'LV' = S_r U''LV'$, $S_{q+1-s} X = P'LQ' = S_p P''LQ'$ and $XY = U''L' = P''L''$ where $L = L'L_1 = L''L_2$. Since the first element of L appears exactly once in XY, it follows that $U_1=P_1=W_1$, $L'=L''$, $U'=S_rW_1$ and $P = S_pW_1$. Hence, $M_t = X_s S_r W_1 LV'Y_s$ and $M_q = X_s S_q W_1 LQ''Y_s$. Next, let: $V' = V''T_i$ and $Q' = Q''T_j$ where $|Q''| < k$ and $|V''| < k$. Then, $XT_{t+1-s} = S_rW_1LV''T_i$, $XT_{q+1-s} = S_pW_1 LQ''T_j$ and $YX = N'V'' = N''Q''$ where $L = N_1 N' = N_2$. Since the last element of L appears exactly once in YX, it follows that $V'' = Q'' = W_2$, $N' = N''$, $V' = W_2T_i$ and $Q' = W_2 T_j$. Hence, $M_t = X_sS_rW_1LW_2T_iY_s$ and $M_q = X_s S_p W_1 LW_2 T_j Y_s$.

In this case, $|L| = n \leq tk + k - sk + |X| = tk + k - sk + sk - k - n = tk - n$, i.e. $2n \leq tk$. □

Definition 5. Let:

$D_1 = D_1(x) = \{(\varphi((0, x, t, i+1)), ..., \varphi((0, x, t, i+n)))|t \geq 1, 0 \leq i < tk - n\} \subseteq D^n$,

$D_2 = \{(\varphi((0, x, t, i+1)), ..., \varphi((0, x, t, i+n+k)))|t \geq 1, 0 \leq i < tk - n + k\} \subseteq D^{n+k}$.

We define a map $f: \{x\} \cup D_1 \to D_2$ by:

$f(x) = ((x, 1, 1), ..., (x, 1, n+k))$, and

$f((\varphi((0, x, t, i+1)), ..., \varphi((0, x, t, i+n))))$

$= (\varphi((0, x, t+1, i+1)), ..., \varphi((0, x, t+1, i+n+k)))$.

Proposition 7. The map f is well defined.

Proof. Let $L = (\varphi((0, x, t, i+1)), ..., \varphi((0, x, t, i+n)))$

$= (\varphi((0, x, q, j+1)), ..., \varphi((0, x, q, j+n)))$

and let $t \leq q$. We have to show that $N = M$ where:

$N = (\varphi((0, x, t+1, i+1)), ..., \varphi((0, x, t+1, i+n+k)))$,

$M = (\varphi((0, x, q+1, j+1)), ..., \varphi((0, x, q+1, j+n+k)))$.

Let $M_t = ULV$, $M_q = PLQ$. Then $M_{t+1} = U_1NV_1$, $M_{q+1} = P_1MQ_1$, such that $|U| = |U_1|$, $|V| = |V_1|$, $|P| = |P_1|$, $|Q| = |Q_1|$. According to the Proposition 6, we have to consider only the three cases: **(b.2), (b.3)** and **(b.4)**, and only for $t \geq s - 1$.

(b.2) $M_t = UL'L_1 V'Y_s$, $M_q = UL'L_1 V'T_{q-t}Y_s$, $UL' = X_s$ and $L = L'L_1$. Then, by the definitions, $M_{t+1} = ULV'YXY_s$, and $M_q = ULV'T_{q-t}YXY_s = ULV'YXT_{q-t}Y_s$. This and the definitions imply that $LV'YX = NR = MR'$ for some R and R'. Since $|N| = |M|$ it follows that $N = M$.

(b.3) $M_t = X_sU'L_2\, L''V$, $M_q = X_s\, S_{q-t}\, U'L_2\, L''V$, $L''V = Y_s$ and $L = L_2\, L''$. Then, by the definitions, $M_{t+1} = X_s\, U'L_2\, YXL''V = X_s\, U'NV$, and $M_q = X_s\, S_{q-t}\, U'L_2\, YXL''V = X_s\, S_{q-t} U'MV$. This implies that $n = L_2YXL'' = M$.

(b.4) $M_t = X_s\, S_r\, W_1\, LW_2\, T_i\, Y_s$ and $M_q = X_s\, S_q\, W_1\, LW_2\, T_jY_s$. Then:

$M_{t+1} = X_s\, S_r\, W_1\, LW_2\, T_i\, YXY_s = X_s\, S_r\, W_1LW_2\, YXT_iY_s = X_s\, S_r\, W_1NV_1$ and

$M_{q+1} = X_s\, S_q\, W_1\, LW_2\, T_j\, YXY_s = X_s\, S_q\, W_1LW_2YXT_j\, Y_s = X_s\, S_q\, W_1\, MQ_1$.

This and the definitions imply that $LW_2YX = NR = MR'$ for some R and R'. Since $|N| = |M|$ it follows that $N = M$. □

It follows directly from Definition 5 that for any $M \in D_2$, $M = f(N)$ for some $N \in D_1$ and if $M = ULV$ where $L \in D^n$ then $L \in D_1 \subseteq D^n$.

Proposition 8. For any $0 \leq p,\ q \leq k$, $(1^p \times f \times 1^{k-p}) \circ f = (1^q \times f \times 1^{k-q}) \circ f$.

Proof. Let $L \in D_1$, and $f(L) = UNV \in D_2$, for $U \in D^p$, $N \in D_1$, $V \in D^{k-p}$. Then, $(1^p \times f \times 1^{k-p}) \circ f(L) = (1^p \times f \times 1^{k-p})(UNV) = Uf(N)V$.

Let

$U = (\varphi((0, x, t, r - p + 1)), ..., \varphi((0, x, t, r)))$,

$N = (\varphi((0, x, t, r + 1)), ..., \varphi((0, x, t, r + n)))$,

$V = (\varphi((0, x, t, r + n + 1)), ..., \varphi((0, x, t, r - p + n + k)))$,

$f(N) = (\varphi((0, x, t + 1, r + 1)), ..., \varphi((0, x, t + 1, r + n + k)))$,

$U_1 = (\varphi((0, x, t + 1, r - p + 1)), ..., \varphi((0, x, t + 1, r)))$, and

$V_1 = (\varphi((0, x, t + 1, r + n + k + 1)), ..., \varphi((0, x, t + 1, r - p + n + 2k)))$.

Then $U_1\, f(N)V_1 = (\varphi((0, x, t + 1, r - p + 1)), ..., \varphi((0, x, t + 1, r - p + n + 2k)))$.

Since $r - p + n + k \leq n + tk$, it follows that $r \leq tk - k + p \leq tk$. This, and Proposition 7, imply that $U = U_1$.

Since $0 \leq r$, it follows that $n < r + n + 1$. This, and Proposition 7, imply that $V = V_1$.

Hence, $U f(N)V = U_1 f(N)V_1$ and since $U_1 f(N)V_1$ does not depend on N it follows that $(1^p \times f \times 1^{k-p}) \circ f(L) = (1^q \times f \times 1^{k-q}) \circ f(L)$ for any $L \in D_1$, i.e. that $(1^p \times f \times 1^{k-p}) \circ f = (1^q \times f \times 1^{k-q}) \circ f$. □

Proposition 9. Let $L = (\varphi((0, x, t, i+1)), ..., \varphi((0, x, t, i+n)))$ and let

$f(L) = (\varphi((0, x, t, i+1)), ..., \varphi((0, x, t, i+n+))) = UNV = PMQ$,

where $N, M \in D^n$. Then $Uf(N)V = Pf(M)Q$.

Proof. It follows from Proposition 8. □

3. CONSTRUCTION 2

Let C be a given nonempty set, $C_1 \subseteq C^n$ and $\rho : C_1 \to C^{n+k}$ be such that:

(1) If $\rho(L) = UNV$ and $|N| = n$, then $N \in C_1$;

(2) If $\rho(L) = UNV = PWQ$ and $|N| = |W| = n$ then $U\rho(N)V = P\rho(W)Q$.

This condition is equivalent to the following: for any $0 \leq p, q \leq k$,

$(1^p \times \rho \times 1^{k-p}) \circ \rho = (1^q \times \rho \times 1^{k-q}) \circ \rho$, where $1^p \times \rho \times 1^{k-p} : \rho(C_1) \to C^{n+2k}$.

For each $x \in C^n \backslash C_1$, let $D(x)$, $E(x)$, $D_1(x)$, φ and f be defined as in the CONSTRUCTION 1.

Definition 6. Let H be the union of C and of all the $D(x)$, $x \in C^n \backslash C_1$. Let $H_1 \subseteq H^n$ be the union of C^n and all the $D_1(x)$, $x \in C^n \backslash C_1$, and let $h: H_1 \to H^{n+k}$ be defined by:

If $x \in C_1$, $h(x) = \rho(x)$;

If $x \in C^n \backslash C_1$, $h(x) = f(x) = ((x, 1, 1), ..., (x, 1, n+k))$;

Proposition 10.

(1) If $h(L) = UNV$ and $|N| = n$, then $N \in H_1$;

(2) If $h(L) = UNV = PWQ$ and $|N| = |W| = n$, then $Uh(N)V = Ph(W)Q$.

This condition is equivalent to the following: for any $0 \leq p, q \leq k$,

$(1^p \times h \times 1^{k-p}) \circ h = (1^q \times h \times 1^{k-q}) \circ h$, where $1^p \times h \times 1^{k-p} : h(H_1) \to H^{n+2k}$.

Proof. Follows from the Definition 6 and CONSTRUCTION 1. □

4. CONSTRUCTION OF FREE (n, $n+k$)-SEMIGROUPS

Let A be a nonempty set. Simply by replacing A with $A\times\{0\}$, we assume that all the new elements introduced in the construction are distinct, and are not elements of A.

Step 0. Let $A_0 = A$, $B_0 = \varnothing \subseteq (A_0)^n$, and $f_0 : B_0 \to (A_0)^{n+k}$ be the empty map. Then the map f_0 satisfies the conditions (1) and (2) from CONSTRUCTION 2**.**

Step 1. We apply CONSTRUCTION 2 for $C = A_0$, $C_1 = B_0$, and $\rho = f_0$, to obtain, H, H_1 and h. Define $A_1 = H$, $B_1 = H_1$ and $f_1 = h : B_1 \to (A_1)^{n+k}$. Then, f_1 satisfies (1) and (2) from CONSTRUCTION 2, and moreover, $A_0 \subseteq A_1$, $B_0 \subseteq (A_0)^n \subseteq B_1 \subseteq (A_1)^n$ and the restriction of f_1 on B_0 is equal to f_0.

Next, we continue by induction. Assume that we have reached

Step m. With this step we have constructed the sets $A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq A_{m-1} \subseteq A_m$, and $B_0 \subseteq (A_0)^n \subseteq B_1 \subseteq (A_1)^n \subseteq \ldots \subseteq B_{m-1} \subseteq (A_{m-1})^n \subseteq B_m \subseteq (A_m)^n$ and the maps $f_j : B_j \to (A_j)^{n+k}$, for $0 \le j \le m$ such that the maps f_j satisfy the conditions (1) and (2) from CONSTRUCTION 2, and the restriction of f_j on B_r is equal to f_r for every $0 \le r < j \le m$.

Step m+1. We apply CONSTRUCTION 2 for $C = A_m$, $C_1 = B_m$, and $\rho = f_m$, to obtain, H, H_1 and h. Define $A_{m+1} = H$, $B_{m+1} = H_1$ and $f_{m+1} = h$. Then, $f_{m+1} : B_{m+1} \to (A_{m+1})^{n+k}$, satisfies the conditions (1) and (2) from CONSTRUCTION 2, and moreover, $A_m \subseteq A_{m+1}$, $B_m \subseteq (A_m)^n \subseteq B_{m+1} \subseteq (A_{m+1})^n$ and the restriction of f_{m+1} on B_m is equal to f_m.

With this procedure, we have constructed sets A_j, B_j and maps $f_j : B_j \to (A_j)^{n+k}$ for $0 \le j$, such that $A_j \subseteq A_{j+1}$, $B_j \subseteq (A_j)^n \subseteq B_{j+1}$, the restriction of f_{j+1} on B_j is equal to f_j and the maps f_j satisfy the conditions (1) and (2) from CONSTRUCTION 2.

Definition 7. Let $F(A) = \bigcup_{0\le j} A_j$, $B= \bigcup_{0\le j} B_j$, $O= \bigcup_{0\le j} (A_j)^{n+k}$ and let f be the union map of all the maps f_j.

Proposition 11. (a) $B = (F(A))^n$;

(b) $O \subseteq (F(A))^{n+k}$; and

(c) $(F(A), f)$ is a free $(n, n + k)$-semigroup with basis A.

Proof. (a) Since for each $0 \le j$, $B_j \subseteq (A_j)^n$, it follows that $B \subseteq F(A)^n$, and since for each $0 \le j$, $A_j \subseteq A_{j+1}$ and $(A_j)^n \subseteq B_{j+1}$, it follows that $F(A)^n \subseteq B$.

(b) Since for each $0 \le j$, $A_j \subseteq F(A)$ it follows that $O \subseteq F(A)^{n+k}$.

(c) From (a), (b), the definition of f and Proposition 10, it follows that $(F(A), f)$ is an $(n, n + k)$-semigroup.

Let (G, g) be an $(n, n + k)$-semigroup, and let $g^t: G^n \to G^{n+tk}$ be the maps as constructed in Propositions 1 and 2. Let $\eta : A \to G$ be a map. We will extend η to an $(n, n + k)$-homomorphism $\psi : F(A) \to G$ in a unique way as follows:

Step 0. Let $\psi_0 = \eta : A_0 \to G$.

Step 1. Let $z \in A_1 = H \equiv$ the union of A_0 and all the $D(x)$, $x \in (A_0)^n \backslash B_0$. If $z \in A_0$, we define $\psi_1(z) = \psi_0(z)$. If $z \notin A_0$, then $z \in D(x)$ for some $x \in (A_0)^n \backslash B_0$, i.e. $z = (x, j, i)$, for some $x \in (A_0)^n \backslash B_0$, $1 \le j \le s$, $1 \le i \le n+sk - jk$. Since $x \in (A_0)^n$, $x = (x_1, x_2, ..., x_n)$ where all the $x_t \in A_0$. Let $y = (\psi_0(x_1), \psi_0(x_2), ..., \psi_0(x_n))$ and let $g^j(y) = (w_1, w_2, ..., w_{n+jk}) \in G^{n+jk}$. In this case we define $\psi_1(z) = w_{jk-k+i}$. This, together with the definition of φ implies that for any t, $\psi_1(\varphi(0, x, j, t)) = w_t$.

We claim that for any $x = (x_1, x_2, ..., x_n) \in B_1$,

$g((\psi_1(x_1), \psi_1(x_2), ..., \psi_1(x_n)) = (\psi_1(a_1), \psi_1(a_2), ..., \psi_1(a_{n+k}))$

where $(a_1, a_2, ..., a_{n+k}) = f_1(x)$.

Proof of the claim. If $x = (x_1, x_2, ..., x_n) \in (A_0)^n$, then $\psi_1(x_t) = \psi_0(x_t)$ and by definition $a_i = (x, 1, i)$, $1 \le i \le n + k$, and $\psi_1(a_i) = w_i$, where

$(w_1, w_2, ..., w_{n+k}) = g^1(y) = g((h_0(x_1), h_0(x_2), ..., h_0(x_n)))$.

If $x \notin (A_0)^n$, then $x \in D_1(u)$ for some $u = (u_1, u_2, ..., u_n) \in (A_0)^n \backslash B_0$, i.e. $x = (\varphi((0, u, j, i + 1)), ..., \varphi((0, u, j, i + n)))$ for some $j \geq 1$, $0 \leq i < tk - n$.

Then, $(\psi_1(x_1), \psi_1(x_2), ..., \psi_1(x_n)) = (w_{jk-k+i+1}, w_{jk-k+i+2}, ..., w_{jk-k+n})$, where

$(w_1, w_2, ..., w_{n+jk}) = g^j((\psi_0(u_1), \psi_0(u_2), ..., \psi_0(u_n))$.

So, $g((\psi_1(x_1), \psi_1(x_2), ..., \psi_1(x_n)) = (v_{jk-k+i+1}, v_{jk-k+i+2}, ..., v_{jk-k+n+k})$, where

$(v_1, v_2, ..., v_{n+jk+k}) = g^{j+1}((\psi_0(u_1), \psi_0(u_2), ..., \psi_0(u_n))$.

Again by the definitions, $f_1(x) = (\varphi((0, u, j + 1, i + 1)), ..., \varphi((0, u, j, i + n + k)))$ and $\psi_1(\varphi(0, u, j + 1, t)) = v_t = \psi_1(a_t)$ for $a_t = \varphi((0, u, j + 1, i + t))$. Hence:

$$g((\psi_1(x_1), \psi_1(x_2), ..., \psi_1(x_n)) = (\psi_1(a_1), \psi_1(a_2), ..., \psi_1(a_{n+k}))$$

where $(a_1, a_2, ..., a_{n+k}) = f_1(x)$.

This implies the claim.

Next we continue by induction. Assume that we have reached Step m.

Step m. With this step we have defined a map $\psi_m : A_m \to G$, such that its restriction to any A_j is equal to ψ_j and such that for any $x = (x_1, x_2, ..., x_n) \in B_m$,

$$g((\psi_m(x_1), \psi_m(x_2), ..., \psi_m(x_n)) = (\psi_m(a_1), \psi_m(a_2), ..., \psi_m(a_{n+k}))$$

where $(a_1, a_2, ..., a_{n+k}) = f_m(x)$.

Step m+1. Let $z \in A_{m+1} = H \equiv A_m$ union all the $D(x)$, $x \in (A_m)^n \setminus B_m$. If $z \in A_m$, we define $\psi_{m+1}(z) = \psi_m(z)$. If $z \notin A_m$, then $z \in D(x)$ for some $x \in (A_m)^n \backslash B_m$, i.e. $z = (x, j, i)$, for some $x \in (A_m)^n \setminus B_m$, $1 \leq j \leq s$, $1 \leq i \leq n + sk - jk$. Since $x \in (A_m)^n$, $x = (x_1, x_2, ..., x_n)$ where all the $x_t \in A_m$. Let $y = (\psi_m(x_1), \psi_m(x_2), ..., \psi_m(x_n))$ and $g^j(y) = (w_1, w_2, ..., w_{n+jk}) \in G^{n+tk}$. We define $\psi_{m+1}(z) = w_{jk-k+i}$.

By the definition, the restriction of ψ_{m+1} to A_m is equal to ψ_m.

We claim that for any $x = (x_1, x_2, ..., x_n) \in B_{m+1}$,

$$g((\psi_{m+1}(x_1), \psi_{m+1}(x_2), ..., \psi_{m+1}(x_n)) = (\psi_{m+1}(a_1), \psi_{m+1}(a_2), ..., \psi_{m+1}(a_{n+k}))$$

where $(a_1, a_2, ..., a_{n+k}) = f_{m+1}(x)$.

Proof of the claim. If $x = (x_1, x_2, ..., x_n) \in B_m \subseteq (A_m)^n$, then by induction and **Step m**: $\psi_{m+1}(x_t) = \psi_m(x_t)$, $f_{m+1}(x) = f_m(x) = (a_1, a_2, ..., a_{n+k})$, and
$g((\psi_{m+1}(x_1), \psi_{m+1}(x_2), ..., \psi_{m+1}(x_n)) = g((\psi_m(x_1), \psi_m(x_2), ..., \psi_m(x_n))$
$= (\psi_m(a_1), \psi_m(a_2), ..., \psi_m(a_{n+k})) = (\psi_{m+1}(a_1), \psi_{m+1}(a_2), ..., \psi_{m+1}(a_{n+k}))$.
All this implies the claim in this case.

If $x = (x_1, x_2, ..., x_n) \in (A_m)^n \backslash B_m$, then $\psi_{m+1}(x_t) = \psi_m(x_t)$ and by definition $a_i = (x, 1, i)$, $1 \le i \le n + k$, and $\psi_{m+1}(a_i) = w_i$, where

$(w_1, w_2, ..., w_{n+k}) = g^1(y) = g((\psi_m(x_1), \psi_m(x_2), ..., \psi_m(x_n)))$.

All this implies the claim in this case.

If $x \notin (A_m)^n$, then $x \in D_1(u)$ for some $u = (u_1, u_2, ..., u_n) \in (A_m)^n \backslash B_m$, i.e. $x = (\varphi((0, u, j, i + 1)), ..., \varphi((0, u, j, i + n)))$ for some $j \ge 1$, $0 \le i < tk - n$. Then,
$(\psi_{m+1}(x_1), \psi_{m+1}(x_2), ..., \psi_{m+1}(x_n)) = (w_{jk-k+i+1}, w_{jk-k+i+2}, ..., w_{jk-k+n})$,
where $(w_1, w_2, ..., w_{n+jk}) = g^j((\psi_m(u_1), \psi_m(u_2), ..., \psi_m(u_n))$. So,
$g((\psi_{m+1}(x_1), \psi_{m+1}(x_2), ..., \psi_{m+1}(x_n)) = (v_{jk-k+i+1}, v_{jk-k+i+2}, ..., v_{jk-k+n+k})$,
where $(v_1, v_2, ..., v_{n+jk+k}) = g^{j+1}((\psi_m(u_1), \psi_m(u_2), ..., \psi_m(u_n))$. By the definitions, $f_{m+1}(x) = (\varphi((0, u, j + 1, i + 1)), ..., \varphi((0, u, j, i + n + k)))$ and
$\psi_{m+1}(\varphi(0, u, j + 1, t)) = v_t = \psi_m(a_t)$ for $a_t = \varphi((0, u, j + 1, i + \mathrm{t}))$. Hence:
$g((\psi_{m+1}(x_1), \psi_{m+1}(x_2), ..., \psi_{m+1}(x_n)) = (\psi_{m+1}(a_1), \psi_{m+1}(a_2), ..., \psi_{m+1}(a_{n+k}))$
where $(a_1, a_2, ..., a_{n+k}) = f_1(x)$.

This implies the claim.

This completes the induction. We define the map ψ to be the union of all the maps ψ_m. From its definition it follows that ψ is an $(n, n{+}k)$-homomorphism and it is unique with these properties.

All this shows that $(F(A), f)$ is a free $(n, n{+}k)$-semigroup with basis A. □

Example 4. Let $n = 1$, $k = 2$.Then $s = 2$, since

$$(s-2)k = (2-2)2 < 1 = n \le (2-1)2 = (s-1)k < 1+2 = n + k \le 4 = sk.$$

Let $A = \{a\}$. Then $D = D(x) = \{(a, 1, 1), (a, 1, 2), (a, 1, 3), (a, 2, 1)\}$ and $D_1 = D = D^n$. So $F(A) = \{a, b, c, d, e\}$, where $b = (a, 1, 1)$, $c = (a, 1, 2)$, $d = (a, 1, 3)$, $e = (a, 2, 1)$, and $f : F(A) \to (F(A))^3$ is defined by:

$$f(a) = (b, c, d),\ \ f(b) = (b, c, e),\ \ f(c) = (c, e, c),\ \ f(d) = (e, c, d),\ \ f(e) = (e, c, e).$$

Р е з и м е

СЛОБОДНИ (*n*, *n* + *k*)-ПОЛУГРУПИ

Во овој труд воведен е поимот за (n, $n + k$)-полугрупи, докажани се некои својства за нив и е даден алгоритамски опис на слободни (n, $n + k$)-полугрупи со дадена база.

Клучни зборови: (n, $n + k$)-полугрупи; слободни (n, $n + k$)-полугрупи

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Received: 1. XII 2009
Accepted: 28. XII 2009