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GENERALIZATION OF A SUMMATION DUE TO RAMANUJAN

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A b s t r a c t: The aim of this research note is to find the sum of the series

$$1+\frac{x-1}{x+1+j}+\frac{(x-1)(x-2)}{(x+1+j)(x+2+j)}+\ldots\quad(\Re\{x\}>0)$$

for j = 0, 1, 2, 3, 4, 5. When j = 0, we get a summation due to Ramanujan. The results are derived with the help of generalized Kummer's theorem obtained already by Lavoie, Grandie and Rathie.

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1. INTRODUCTION

We start with an interesting summation due to Ramanujan [4], *viz*.

$$1+\frac{x-1}{x+1}+\frac{(x-1)(x-2)}{(x+1)(x+2)}+\ldots=\frac{2^{2x-1}\Gamma^2(x+1)}{\Gamma(2x+1)}\qquad(\Re\{x\}>0).\tag{1}$$

As pointed out by Berndt [2], this summation can be obtained quite simply by employing Kummer's summation theorem [1] *viz*.

$${}_2F_1\left[\begin{matrix}a,b\\1+a-b\end{matrix};-1\right]=\frac{\sqrt{\pi}\,\Gamma(1+a-b)}{2^a\,\Gamma\left(1+\frac{a}{2}-b\right)\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)}\tag{2}$$

by taking a = 1 and $b = 1 - x$.

In 1996 Lavoie, Grandie and Rathie [3] have obtained explicite expressions of

$$ {}_2F_1\left[\begin{matrix} a, b \\ 1+a-b+j \end{matrix} ;-1\right] \tag{3}$$

for j = 0, ±1, ±2, ±3, ±4, ±5. The case j = 0 was presented above in (2) as the Kummer's theorem. However, the following another special cases with non–negative j will be required in our present investigations:

$$ {}_2F_1\left[\begin{matrix} a, b \\ 1+a-b+j \end{matrix} ;-1\right] = \frac{\sqrt{\pi}\,\Gamma(1+a-b+j)\,\Gamma(1-b)}{2^a\,\Gamma\left(1-b+\frac{1}{2}(j=|j|)\right)} $$
$$ \times\left[\frac{A_j}{\Gamma\left(\frac{a}{2}-b+\frac{j}{2}+1\right)\Gamma\left(\frac{a}{2}+\frac{j}{2}+\frac{1}{2}-\left[\frac{j+1}{2}\right]\right)}\right. $$
$$ \left.+\frac{B_j}{\Gamma\left(\frac{a}{2}-b+\frac{j}{2}+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}+\frac{1}{2}-\left[\frac{j}{2}\right]\right)}\right] \tag{4}$$

where, as usual, [x] denotes the greatest integer less then or equal to x and the values of the constants A_j, B_j are given in the Table 1. below.

Table 1

j	A_j	B_j
0	1	0
1	–1	1
2	$a - b + 1$	–2
3	$3b - 2a - 5$	$2a - b + 1$
4	$2(a - b + 3)(1 + a - b) - (b - 1)(b - 4)$	$-4(a - b + 2)$
5	$-4(6 + a - b)^2 + 2(b + 11)$ $\times (6 + a - b) + b^2 - 13b - 20$	$4(6 + a - b)^2 + 2(b - 17)$ $\times (6 + a - b) - b^2 - b + 62$

The aim of this note is to find an interesting generalization of the Ramanujan's summation (1) by making sense of (4). Five summations closely related to Ramanujan's summation have also been obtained as special cases of our main findings. The summations derived here are simple, interesting easily established and may be useful.

2. MAIN SUMMATION

Theorem 1.

$$1+\frac{x-1}{x+1+j}+\frac{(x-1)(x-2)}{(x+1+j)(x+2+j)}+\ldots$$

$$=\frac{\sqrt{\pi}(1+j)\,\Gamma(x)}{2}\left[\frac{A_j}{\Gamma\left(x+\frac{1}{2}+\frac{j}{2}\right)\Gamma\left(1+\frac{j}{2}-\left[\frac{1+j}{2}\right]\right)}\right.$$

$$\left.+\frac{B_j}{\Gamma\left(x+\frac{j}{2}\right)\Gamma\left(\frac{1}{2}+\frac{j}{2}-\left[\frac{j}{2}\right]\right)}\right] \quad (5)$$

for $j = 0, 1, 2, 3, 4, 5$. *The coefficients* A_j, B_j *can be obtained from the table by changing* a *by* 1 *and* b *by* $1-x$ *respectively*.

Proof. In (4) taking $a = 1$ and $b = 1 - x$, then expressing the hypergeometric function as a series, we have

$${}_2F_1\left[\begin{matrix}1, 1-x \\ 1+x+j\end{matrix};-1\right]=\sum_{n=0}^{\infty}\frac{(1)_n(1-x)_n}{(1+x+j)_n}\,\frac{(-1)^n}{n!}$$

$$=1+\frac{x-1}{x+1+j}+\frac{(x-1)(x-2)}{(x+1+j)(x+2+j)}+\ldots \quad (6)$$

Here $(a)_0 := 1$, $(a)_n = a(a-1)\dots(a-n+1)$, $n \in \mathbb{N}$ stands for the *Pochhammer-symbol*, called sometimes *shifted factorial* as well.

Similarly, putting $a = 1$, $b = 1 - x$ on the right–hand expression in (4), we get

$$\frac{\Gamma\left(\frac{3}{2}\right)\Gamma(x)\Gamma(1+x+j)}{\Gamma(x+j)}\left[\frac{A_j}{\Gamma\left(x+\frac{1}{2}+\frac{j}{2}\right)\Gamma\left(1+\frac{j}{2}-\left[\frac{1+j}{2}\right]\right)}\right.$$

$$\left.+\frac{B_j}{\Gamma\left(x+\frac{j}{2}\right)\Gamma\left(\frac{1}{2}+\frac{j}{2}-\left[\frac{j}{2}\right]\right)}\right] \tag{7}$$

such that, after easy simplification, one tranforms into the asserted right–hand expression of (5). This completes the derivation of (5). □

3. SPECIAL CASES

In (5), if we take $j = 0, 1, 2, 3, 4, 5$ we have the following interesting summations.

1. For $j = 0$

$$1+\frac{x-1}{x+1}+\frac{(x-1)(x-2)}{(x+1)(x+2)}+\dots=\frac{\Gamma\left(\frac{3}{2}\right)\Gamma(1+x)}{\Gamma\left(x+\frac{1}{2}\right)} \tag{8}$$

The right–hand side of (8) can be seen equivalent to the right–hand side of (5).

2. For $j = 1$

$$1+\frac{x-1}{x+6}+\frac{(x-1)(x-2)}{(x+6)(x+7)}+\ldots$$

$$=(1+x)\,\Gamma(x)\Gamma\left(\frac{3}{2}\right)\left[\frac{1}{\Gamma\left(x+\frac{1}{2}\right)}-\frac{1}{\Gamma\left(\frac{1}{2}\right)\Gamma(x+1)}\right]. \quad (9)$$

3. For $j = 2$

$$1+\frac{x-1}{x+3}+\frac{(x-1)(x-2)}{(x+3)(x+4)}+\ldots$$

$$=(x+2)\,\Gamma(x)\Gamma\left(\frac{3}{2}\right)\left[\frac{1+x}{\Gamma\left(x+\frac{3}{2}\right)}-\frac{2}{\Gamma\left(\frac{1}{2}\right)\Gamma(x+1)}\right]. \quad (10)$$

4. For $j = 3$

$$1+\frac{x-1}{x+4}+\frac{(x-1)(x-2)}{(x+4)(x+5)}+\ldots$$

$$=(x+3)\,\Gamma(x)\Gamma\left(\frac{3}{2}\right)\left[\frac{x+2}{\Gamma\left(x+\frac{3}{2}\right)}-\frac{3x+4}{\Gamma\left(\frac{1}{2}\right)\Gamma(x+2)}\right]. \quad (11)$$

5. For $j = 4$

$$1+\frac{x-1}{x+5}+\frac{(x-1)(x-2)}{(x+5)(x+6)}+\ldots$$

$$=(x+2)\,(x+4)\Gamma(x)\Gamma\left(\frac{3}{2}\right)\left[\frac{x+3}{\Gamma\left(x+\frac{5}{2}\right)}-\frac{4}{\Gamma\left(\frac{1}{2}\right)\Gamma(x+2)}\right]. \quad (12)$$

6. For $j = 5$

$$1+\frac{x-1}{x+6}+\frac{(x-1)(x-2)}{(x+6)(x+7)}+\ldots$$

$$=(x+5)\,\Gamma(x)\Gamma\left(\frac{3}{2}\right)\left[\frac{x^2+7x+12}{\Gamma\left(x+\frac{5}{2}\right)}-\frac{5x^2+25x+32}{\Gamma\left(\frac{1}{2}\right)\Gamma(x+3)}\right]. \qquad (13)$$

Clearly, (8) is a Ramanujan's summation and other summations (9) to (13) are seen to be closely related (8).

Remark 1. *For another summations due to Ramanujan and their generalizations the interested reader can consult* [5].

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REFERENCES

[1] W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge University Press, Cambridge, 1935.

[2] B. C. Berndt, *Ramanujan's Notebooks*, Part II, Springer–Verlag, New York, 1985.

[3] J. L. Lavoie, F. Grondin, A. K. Rathie, Generalizations of Whipple's theorem on the sum of a ${}_3F_2$, *J. Comput. Appl. Math.*, **72** (1996), 293–300.

[4] S. Ramanujan, *Notebooks*, Volume 2., Tata Institute of Fundamental Research, Bombay, 1957.

[5] A. K. Rathie, S. Malani, R. Mathur, J. Choi, Certain summations due to Ramanujan and their generalizations, *Bull. Korean Math. Soc.*, **42** (3) (2005), 469–475.

Р е з и м е

ГЕНЕРАЛИЗАЦИЈА НА СУМИРАЊЕТО НА RAMANUJAN

Целта на ова истражување е да се најде сумата на редот

$$1+\frac{x-1}{x+1+j}+\frac{(x-1)(x-2)}{(x+1+j)(x+2+j)}+\ldots \quad (\Re\{x\}>0)$$

за $j = 0, 1, 2, 3, 4, 5$. За $j = 0$, ја добиваме сумата на Ramanujan. Резултатите се добиени со помош на генерализираната теорема на Kummer, дадена од Lavoie, Grandie и Rathie.

Клучни зборови: хипергеометриска ${}_2F_1$; сумирање на теоремата на Kummer; сумациона формула на Ramanujan

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