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SOME WEAKER FORMS OF SMOOTH FUZZY CONTINUOUS FUNCTIONS

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In this paper we introduce various notions of continuous fuzzy proper functions by using the existing notions of fuzzy closure and fuzzy interior operators like R_τ^r-closure, R_τ^r-interior, etc., and present all possible relations among these types of continuities. Next, we introduce the concepts of α-quasi-coincidence, q_α^r-pre-neighborhood, q_α^r-pre-closure and q_α^r- pre-continuous function in smooth fuzzy topological spaces and investigate the equivalent conditions of q_α^r- pre-continuity.

Key words: fuzzy proper function; smooth fuzzy topology; smooth fuzzy continuity; fuzzy closure; fuzzy interior

INTRODUCTION

Šostak [28] defined *I*-fuzzy topology as an extension of Chang's fuzzy topology [2]. It has been developed in many directions by many authors. For example see [8, 16]. Ramadan [23] gave a similar definition of fuzzy topology on a fuzzy set in Šostak's sense and called by the name "*smooth fuzzy topological space*".

On the other hand, studying different forms of continuous functions in topological space is an interesting area of research which attracts many researchers. In the fuzzy context, after the introduction of fuzzy proper function from a fuzzy set in to a fuzzy set [1], several notions of continuous fuzzy proper functions between Chang's fuzzy topological spaces are defined and their properties are discussed in [3]. The concepts of smooth fuzzy continuity, weakly smooth fuzzy continuity, *qn*-weakly smooth fuzzy continuity, (α,β)-weakly smooth fuzzy continuity of a fuzzy proper function on smooth fuzzy topological spaces and their inter-relations are investigated in [5, 23, 26, 27, 10].

Lee and Lee [19] introduced the notion of fuzzy *r*-interior which is an extension of Chang's fuzzy interior. Using fuzzy *r*-interior, they define fuzzy *r*-semiopen sets and fuzzy *r*-semicontinuous maps which generalize fuzzy semiopen sets and fuzzy semicontinuous maps in Chang's fuzzy topology, respectively. Some basic properties of fuzzy *r*-semiopen sets and fuzzy *r*-semicontinuous maps are investigated in [19]. In [22], the concepts of several types of weak smooth compactness are introduced and investigated some of their properties.

In [7, 20], the notions of fuzzy semicontinuity, fuzzy γ-continuity of a fuzzy proper functions, fuzzy separation axioms, fuzzy connectedness and fuzzy compactness are defined.

Ganguly and Saha [6] introduced the notions of δ-cluster points and θ-cluster points in Chang's fuzzy topological spaces. Kim and Park [15] introduced δ-closure in Šostak's fuzzy topological spaces. Kim and Ko [13] introduced fuzzy super continuity, fuzzy δ-continuity, fuzzy almost continuity in the context of Šostak's fuzzy topological spaces. They proved that fuzzy super continuity implies both fuzzy δ-continuity and fuzzy almost continuity. Similar works are discussed by various researchers, see [12, 14, 18, 21].

By using the existing notions of fuzzy closure and fuzzy interior operators, we introduce the concepts of fuzzy weakly δ-continuity, fuzzy weakly δ-r_1 -continuity, fuzzy weakly δ-$[r,q]_1$-continuity, fuzzy weakly δ-r_2-continuity, fuzzy weakly δ-$[r,q]_2$-continuity, fuzzy weakly δ-r_3-continuity, fuzzy weakly δ-r_4-continuity, fuzzy almost r_1-continuity, fuzzy almost $[r,q]_1$- continuity, fuzzy almost r_2-continuity, fuzzy almost $[r,q]_2$-continuity, fuzzy almost r_3-continuity and fuzzy almost r_4-continuity and discuss the inter-relations among them.

Further, by introducing the notions α-quasi-coincidence, q_α^r-pre-neighborhood, q_α^r-pre-closure and q_α^r-pre-continuity, we investigate the relations between q_α^r - pre-continuity and the property $F(PCl_a(A, r)) \leq PCl_a(F(A),r)$, for every $A \leq \mu$ in smooth fuzzy topological spaces.

PRELIMINARIES

Let X, S be non-empty sets. We denote by I, I_0, I^X, 0_X, μ and ν respectively the unit interval [0, 1], the interval [0, 1], the set of all fuzzy subsets of X, the zero function on X, a fixed fuzzy subset of X and a fixed fuzzy subset of S. For $X=\{x_1, x_2, \ldots, x_n\}$ and $\lambda_i \in I$, $i \in \{1,2,\ldots,n\}$, we denote the fuzzy subset μ of X which maps x_i to λ_i for every $i = 1,2,\ldots,n$ by $\mu\frac{[\lambda_1,\lambda_2,\ldots,\lambda_n]}{[x_1, x_2,\ldots, x_n]}$. A fuzzy point [15] in X is defined by $P_x^\lambda(t) = \{\begin{smallmatrix}\lambda & if & t=x \\ 0 & if & t\neq x\end{smallmatrix}$, where $0 < \lambda \leq 1$. By $P_x^\lambda \in \mu$ we mean that $\lambda \leq \mu(x)$.

Definition 1 [23]: *A smooth fuzzy topology on a fuzzy set $\mu \in I^X$ is a map $\tau : \mathcal{J}_\mu = \{U \in I^X: U \leq \mu\} \to I$, satisfying the following axioms:*
1. $\tau(0_X) = \tau(\mu) = 1$,
2. $\tau(A_1 \wedge A_2) \geq \tau(A_1) \wedge \tau(A_2), \forall A_1, A_2 \in \mathcal{J}_\mu$,
3. $\tau(\vee_{i\in\Gamma} A_i) \geq \wedge_{i\in\Gamma} \tau(A_i)$ *for every family* $(A_i)_{i\in\Gamma} \subseteq \mathcal{J}_\mu$.

The pair (μ,τ) is called a smooth fuzzy topological space.

A fuzzy subset $U \in \mathcal{J}_\mu$ is said to be fuzzy open if $\tau(U) > 0$ and fuzzy closed if $\tau(\mu - U) > 0$.

Definition 2 [1]: *Let $U, V \in \mathcal{J}_\mu$ are said to be quasi-coincident referred to μ (written as $UqV[\mu]$) if there exists $x \in X$ such that $U(x)+V(x)>\mu(x)$. If U is not quasi-coincident with V, then we write, $U\bar{q}V[\mu]$.*

A fuzzy set $U \in \mathcal{J}_\mu$ is called a q-neighborhood of a fuzzy point P_x^λ in μ if $P_x^\lambda qU[\mu]$ and $\tau(U) > 0$.

Definition 3[1]: *Let $\mu \in I^X$ and $\nu \in I^S$. A non-zero fuzzy subset F of $X \in S$ is said to be a fuzzy proper function from μ to ν if*

1. $F(x,s) \leq min\{\mu(x), \nu(s)\}, \forall (x,s) \in X \times S$,
2. *for each $x \in X$ with $\mu(x)>0$, there exists a unique $s_0 \in S$ such that $F(x, s_0)= \mu(x)$ and $F(x, s) = 0$ if $s\neq s_0$.*

Definition 4 [1]: *Let F be a fuzzy proper function from μ to ν. If $U \in \mathcal{J}_\mu$ and $V\in \mathcal{J}_\mu$, then $F(U):S \to I$ and $F^{-1}(V) : X \to I$ are defined by*

$$(F(U))(s)=\sup \{F(x, s) \wedge U(x) : x \in X\}, \forall s \in S,$$

$$(F^{-1}(V))(x) = \sup \{F(x, s) \wedge V(s) : s \in S\}, \forall x \in X.$$

The inverse image of a fuzzy subset V under a fuzzy proper function F can be easily obtained as $(F^{-1}(V))(x) = \mu(x) \wedge V(s)$, where $s \in S$ is the unique element such that $F(x,s)=\mu(x)$.

Definition 5 [5]: *A fuzzy proper function $F:\mu\to \nu$ is said to be injective (or one-to-one) if $F(x_1, s) > 0$ and $F(x_2, s) > 0$, for some $x_1, x_2 \in X$ and $s \in S$, then $x_1 = x_2$.*

Definition 6 [4]: *Let (μ, τ) be a smooth fuzzy topological space. For $r \in I_0, A \in \mathcal{J}_\mu$,*
- $C_\tau: \mathcal{J}_\mu \times I_0 \to \mathcal{J}_\mu$ *is defined by* $C_\tau(A, r) = \wedge\{K \in \mathcal{J}_\mu : A \leq K, \tau(\mu - K) \geq r\}$,
- $I_\tau : \mathcal{J}_\mu \times I_0 \to \mathcal{J}_\mu$ *is defined by* $I_\tau(A,r) = \vee \{ S \in \mathcal{J}_\mu: S \leq A, \tau(S) \geq r\}$.

Definition 7 (Cf. [18]): *Let (μ,τ) be smooth fuzzy topological space, $U \in \mathcal{J}_\mu$, and $r \in I_0$. Then*
- *U is called fuzzy r-preopen if $U \leq I_\tau(C_\tau(U,r),r)$,*
- *U is called fuzzy r-preclosed if $U \geq C_\tau(I_\tau(U,r),r)$.*

Definition 8 [13]: *Let (μ,τ) be a smooth fuzzy topological space and let $A \in \mathcal{J}_\mu$, $r \in I_0$. Then,*

- *A is called a Q_τ^r-neighborhood of P_x^λ if $P_x^\lambda qA[\mu]$ with $\tau(A) \geq r$,*
- *A is called a R_τ^r-neighborhood of P_x^λ if $P_x^\lambda qA[\mu]$ with $A = I_\tau(C_\tau(A,r),r)$.*

Definition 9 [11]: *Let (μ,τ) be a smooth fuzzy topological space and let $A \in \mathcal{J}_\mu$, $r \in I_0$. Then, we define,*
- *Smooth fuzzy R_τ^r-closure of A by*

$\mathbb{D}_\tau(A, r) = \vee\{P_x^\lambda \in \mu : C_\tau(U,r)qA[\mu], \forall R_\tau^r -$ *neighborhood U of P_x^λ*$\}$.
- *Smooth fuzzy R_τ^r-interior of A by*

$\mathbb{I}_\tau(A, r) = \vee\{K \in \mathcal{J}_\mu : A \geq C_\tau(K,r), K = I_\tau(C_\tau(K,r),r)\}$.

Theorem 1 [11]: *Let (μ,τ) be a smooth fuzzy topological space. For $A \in \mathcal{J}_\mu$ and $r \in I_0$, then*

$$\mathbb{D}\tau(A,r) \wedge\{K \in \mathcal{J}_\mu: A \leq I_\tau(K,r), K = C_\tau(I_\tau(K,r),r)\}.$$

Definition 10 (Cf. [13]): *Let (μ,τ) and (ν,σ) be two smooth fuzzy topological spaces and $F:\mu \to \nu$ be a fuzzy proper function. Then, F is called fuzzy almost continuous or FAC if for every R_σ^r-neighborhood V of $F(P_x^\lambda)$, there exists an Q_τ^r-neighborhood U of P_x^λ such that $F(U) \leq V$.*

Theorem 2 [9]: *Let $F:\mu \to \nu$ be a fuzzy proper function such that $\nu = F(\mu)$. If F is one-to-one, then $F^{-1}(\nu - V) = \mu - F^{-1}(V)$, $\forall V \in \mathcal{J}_\mu$.*

FUZZY WEAKLY δ-CONTINUOUS AND FUZZY ALMOST CONTINUOUS FUNCTIONS

Definition 11: *Let (μ,τ) and (ν,σ) be smooth fuzzy topological spaces, $F:\mu \to \nu$be a fuzzy proper function and $r, q \in I_0$ be fixed. Then, F is called*

(1) fuzzy weakly δ-continuous or FW δ-C if for every R_σ^r-neighborhood V of $F(P_x^\lambda)$, there exists an R_τ^r-neighborhood U of P_x^λ such that $F(C\tau(U,r)) \leq V$,

(2) fuzzy weakly δ-r_1-continuous or FW δ-r_1-C if
$$F(\mathbb{D}_\tau(A, r)) \leq \mathbb{D}\sigma(F(A), r),\ \forall A \in \mathcal{J}_\mu,$$

(3) fuzzy weakly δ-$[r,q]_1$-continuityor FW δ-$[r,q]_1$-C if
$$F(\mathbb{D}_\tau(A, r)) \leq \mathbb{D}\sigma(F(A), q),\ \forall A \in \mathcal{J}_\mu,$$

(4) fuzzy weakly δ-r_2-continuous or FW δ-r_2-C if
$$\mathbb{D}_\tau(F^{-1}(V), r) \leq F^{-1}(\mathbb{D}_\sigma(V,r)),\ \forall V \in ,$$

(5) fuzzy weakly δ-$[r,q]_2$-continuous or FW δ-$[r,q]_1$-C if
$$\mathbb{D}_\tau(F^{-1}(V), r) \leq F^{-1}(\mathbb{D}_\sigma(V, q)),\ \forall V \in \mathcal{J}_\mu,$$

(6) fuzzy weakly δ-r_3-continuous or FW δ-r_3-C if
$$\mathbb{D}_\tau(F^{-1}(V), r) = F^{-1}(V),\ \forall V \in \mathcal{J}_\mu \text{ with } V= \mathbb{D}_\sigma(V,r),$$

(7) fuzzy weakly δ-r_4-continuous or FW δ-r_4-C if
$$\mathbb{D}_\tau(\mu - F^{-1}(V), r) = \mu - F^{-1}(V),\ \forall V \in \mathcal{J}_\mu \text{ with } V= \mathbb{I}_\sigma(V, r).$$

Theorem 3 *Let $F : (\mu,\tau) \to (\nu,\sigma)$ be a one-to-one fuzzy proper function with $\nu=F(\mu)$. If F is fuzzy weakly δ-continuous, then F is fuzzy weakly δ-r_1-continuous*

Proof. Suppose that there exist $A \in \mathcal{J}_\mu$ and $r \in I_0$ such that

$$F(\mathbb{D}_\tau(A, r))(s) > \mathbb{D}_\sigma(F(A), r)(s),$$

for some $s \in S$. Then, there exists $x \in X$ such that $F(x,s) > 0$. Since F is one-to-one and $F(\mu) = \nu$, we have $F(\mathbb{D}_\tau(A, r))(s) = \mathbb{D}_\tau(A, r))(x) > \mathbb{D}_\sigma(F(A), r)(s)$. Now we choose a real number η such that $\mathbb{D}_\tau(A, r))(x) > \eta > \mathbb{D}_\sigma(F(A), r)(s)$. Since $P_s^\eta \notin \mathbb{D}_\sigma(F(A), r)$, there exists an R_σ^r-neighborhood V of $F(P_x^\eta) = P_s^\eta$ such that $C_\tau(V,r)\bar{q}F(A)[V]$ which implies that $F(A) \leq \nu - C_\tau(V,r)$. Since F is fuzzy weakly δ-continuous, there exists an R_τ^r-neighborhood U of P_x^η such that $F(C_\tau(U,r) \leq V \leq C_\tau(V,r)$. Thus, $F(A) \leq \nu - F(C_\tau(U,r))$. Using the facts that F is one-to-one and $F(\mu)=\nu$ and using Theorem 2, we get

$$A \leq F^{-1}(F(A)) \leq F^{-1}(\nu - F(C_\tau(U,r))) = \mu - F^{-1}(F(C_\tau(U,r))) \leq \mu - C_\tau(U,r).$$

Therefore, $A\ \bar{q}C_\tau(U,r)[\mu]$ and $P_x^\eta \notin \mathbb{D}_\tau(A),r)$ which implies that $P_s^\eta = F(P_x^\eta) \notin F(\mathbb{D}_\tau(A,r))$, which is a contradiction to $F(\mathbb{D}_\tau(A,r)) > \eta$. Hence, it follows that $F(\mathbb{D}_\tau(A,r)) \leq \mathbb{D}_\sigma(F(A),r)$.

The statement of the above theorem is not true when F is not one-to-one or $F(\mu) \neq \nu$. The following counterexamples justify our statement.

Counterexample 1: Let $X = \{x,y\}, S = \{s,t\}, \mu\left\{\begin{smallmatrix}[0.8,0.7]\\ [x,y]\end{smallmatrix}\right. \in I^X, \nu\begin{smallmatrix}[0.8,0]\\ [s,t]\end{smallmatrix} \in I^S$, $U_1\begin{smallmatrix}[0.4,0.3]\\ [x,y]\end{smallmatrix} \in \mathcal{J}_\mu$ and $V_1\begin{smallmatrix}[0.4,0]\\ [s,t]\end{smallmatrix} \in \mathcal{J}_\nu$. We define $\tau : \mathcal{J}_\mu \to I$ and $\sigma : \mathcal{J}_\nu \to I$ by

$$\tau(U) = \begin{cases} 1, & U = 0_X \text{ , or } \mu \\ 0.7, & U = U_1\text{ ,} \\ 0, & \text{otherwise} \end{cases}$$

and

$$\sigma(V) = \begin{cases} 1, & V = 0_S \text{ or } \nu \\ 0.6, & V = V_1, \\ 0, & \text{otherwise.} \end{cases}$$

If the fuzzy proper function $F : (\mu, \tau) \to (\nu,\sigma)$ is defined by

$$F(x,s)=0.8,\ F(x,t) = 0,\ F(y,s) = 0.7,\ F(y,t) = 0,$$

Then F is not one-to-one and $F(\mu)\begin{smallmatrix}[0.8,0]\\ [s,t]\end{smallmatrix} = \nu$. We fix $r = 0.5$. For $P_l^\eta \in \mu$ and for the R_σ^r-neighborhood V_1 of $F(P_l^\eta)$, we can choose U_1 as an R_τ^r-neighborhood of P_l^λ satisfying $F(C_\tau(U_1,r))\begin{smallmatrix}[0.4,0]\\ [s,t]\end{smallmatrix} = V_1$. For ν we find μ such that $F(C_\tau(\mu,r)) = \nu$. Thus F is fuzzy weakly δ-continuous.

Consider the fuzzy point $P_y^{0.45} \in \mu$ and the fuzzy set $A_{[x,y]}^{[0.04]} \in \mathcal{J}_\mu$. If $U \in \mathcal{J}_\mu$ is such that $U = I_\tau(C_\tau(U,r),r)$, then $U = 0_X$ or μ or U_1 and both are the R_τ^r ods $P_y^{0.45}$. Here, $C_\tau(U_1,r), qA[\mu]$ and $C_\tau(\mu,r), qA[\mu]$. Therefore, $P_y^{0.45} \in \mathbb{D}_\tau(A,r)$ and $F(P_y^{0.45}) = P_s^{0.45} \in F(\mathbb{D}_\tau(A,r)$. Since, $V_1(s) + 0.45 > 0.8 = \nu(s)$ and $I_\sigma(C_\sigma(V_1,r),r) = C_\sigma\left((\nu - V_1)\begin{smallmatrix}[0.4,0]\\ [s,t]\end{smallmatrix}, r\right) = V_1,$

V_1 is R_τ^r-neighborhood of $P_s^{0.45}$. We note that F(A) $\begin{smallmatrix}[0.4,0]\\ [s,t]\end{smallmatrix}$ and $F(A)$ $\bar{q}C_\sigma(V,r)[\nu]$ and hence $P_s^{0.45} \notin \mathbb{D}_\sigma(F(A),r)$. Therefore F is not fuzzy weakly δ-r_1-continuous.

Counterexample 2: Let $X = \{x,y\}, S = \{s,t\}$, $\mu\{\begin{smallmatrix}[0.9,0.8]\\ x,y\end{smallmatrix} \in I^X, \nu\begin{smallmatrix}[1,1,]\\ [s,t]\end{smallmatrix} \in I^S$, $U_1\begin{smallmatrix}[0.4,0.3]\\ [x,y]\end{smallmatrix} \in \mathcal{J}_\mu$ and $V_1\begin{smallmatrix}[0.5,0.5]\\ [s,t]\end{smallmatrix} \in \mathcal{J}_\nu$.

We define $\tau : \mathcal{J}_\mu \to I$ and $\sigma : \mathcal{J}_\nu \to I$ by

$$\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu \\ 0.6, & U = U_1, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\sigma(V) = \begin{cases} 1, & V = 0_S \text{ or } \nu \\ 0.5, & V = V_1, \\ 0, & \text{otherwise.} \end{cases}$$

Let the fuzzy proper function $F : (\mu,\tau) \to (\nu,\sigma)$ be defined by

$F(x,s) = 0.9, F(x,t) = 0, F(y,s) = 0, F(y,t) = 0.8.$

Then, $F(\mu)\begin{smallmatrix}[0.9,0.8]\\ [s,t]\end{smallmatrix} \neq \nu$. If $r = 0.5$, and for the R_σ^r-neighborhood V_1 of $F(P_l^\eta)$, we can fine U_1 as a required R_τ^r-neighborhood of $P_l^\eta \in \mu$. Indeed, we first note that $F(C_\tau(U_1,r))\begin{smallmatrix}[0.5,0.5]\\ [s,t]\end{smallmatrix} = V_1$. Since the only R_σ^r-neighborhoods of $F(P_l^\eta)$ are V_1 and ν, it follows that F is fuzzy weakly δ-continuous.

Consider $P_y^{0.55} \in \mu$ and $A\begin{smallmatrix}[0.04]\\ [x,y]\end{smallmatrix} \in \mathcal{J}_\mu$. Since $P_y^{0.55} q U_1[\mu]$ and $P_y^{0.55} q\mu[\mu]$, U_1 and μ are the R_τ^r -neighborhoods of $P_y^{0.55}$. Since $C_\tau(U_1,r), qA[\mu]$ and $C_\tau(\mu,r), qA[\mu]$, we have $P_y^{0.55} \in \mathbb{D}_\tau(A,r)$ and $F(P_y^{0.55}) = P_t^{0.55} \in \mathbb{D}_\tau(A,r)$. Using

$V_1(t) + 0.55 > 1 = \nu(t)$ and

$$I_\sigma(C_\sigma(V_1,r),r) = I_\sigma\left((\nu - V_1)\begin{smallmatrix}[0.5,0.5]\\ [s,t]\end{smallmatrix}, r\right) = V_1,$$

we get that V_1 is an R_τ^r-neighborhood of $P_t^{0.55}$. But F $(A)\begin{smallmatrix}[0.4,0]\\ [s,t]\end{smallmatrix} \bar{q}C_\sigma(V,r)\,[\nu]$ implies that $P_t^{0.55} \notin \mathbb{D}_\sigma(F(A),r)$.

Theorem 4: *Let F: $(\mu,\tau) \to (\nu,\sigma)$ be a fuzzy proper function. If (a) F is fuzzy weakly δ-r_1-continuous, (b) F is fuzzy weakly δ-r_2-continuous, (c) F is fuzzy weakly δ-r_3-continuous, then (a)⇒ b)⇒ (c).*

Proof is straightforward.

Theorem 5: *Let F: $(\mu,\tau) \to (\nu,\sigma)$ be a one-to-one fuzzy proper function with ν=$F(\mu)$. If F is fuzzy weakly δ-r_3-continuous, then F is δ-r_4-continuous.*

Proof. Let $V \in \mathcal{J}_\nu$ with $V = \mathbb{I}_\sigma(V,r)$. Then, $\nu - V = \nu - \mathbb{I}_\sigma(V,r) = \mathbb{D}_\sigma(\nu - V, r)$. By using the hypothesis, we get $\mathbb{D}_\tau(F^{-1}(\nu - V), r) = F^{-1}(\nu - V)$. Since F is one-to-one and $\nu = F(\mu)$ and by Theorem 2, we have $F^{-1}(\nu - V) = \mu - F^{-1}(V)$. Therefore, $\mathbb{D}_\tau(\mu - F^{-1}(V), r) = \mu - F^{-1}(V)$. □

The statement of the above theorem is not true when F is not one-to-one or $F(\mu) \neq \nu$. The following counterexamples justify our statement.

Counterexample 3: Let $X = \{x,y\}, S = \{s,t\}$,. We define $\mu\{\begin{smallmatrix}[0.8,0.6]\\ [x,y]\end{smallmatrix} \in I^X$, $\nu\begin{smallmatrix}[0.8,0]\\ [s,t]\end{smallmatrix} \in I^S$, $U_n\begin{smallmatrix}[0.4+\frac{1}{n+10},0.4+\frac{1}{n+10}]\\ [x,y]\end{smallmatrix}$, where n = 1, 2,… and $V_1\begin{smallmatrix}[0.4,0]\\ [s,t]\end{smallmatrix} \in \mathcal{J}_\nu$. If $\tau : \mathcal{J}_\mu \to I$ and $\sigma : \mathcal{J}_\nu \to I$ are defined by

$$\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu \\ 0.6, & U = U_n \forall n \quad \text{or} \quad \bigvee U_n, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\sigma(V) = \begin{cases} 1, & V = 0_S \text{ or } \nu \\ 0.4, & U = V_1, \\ 0, & \text{otherwise} \end{cases}$$

then (μ,τ) and (ν,σ) are smooth fuzzy topological spaces. Let the fuzzy proper function $F : (\mu,\tau) \to (\nu,\sigma)$ be defined by

$F(x,s) = 0.8, F(x,t) = 0, F(y,s) = 0.6, F(y,t) = 0.$

We fix $r = 0.4$. Since $C_\sigma(V_1,r) = V_1 = I_\sigma(V_1,r)$ and $C_\tau(U_n,r) = U_n = I_\tau(U_n,r)$, n = 1, 2, …, we get $\mathbb{D}_\sigma(V_1,r) = V_1$, $F^{-1}(V_1)\begin{smallmatrix}[0.4,0.4]\\ x,y\end{smallmatrix} \leq I_\tau(U_n,r)$, and $C_\tau(I_\tau(U_n,r),r) = U_n$. Therefore, $\mathbb{D}_\tau(F^{-1}(V_1),r) = (\bigwedge U_n)\begin{smallmatrix}[0.4,0.4]\\ [x,y]\end{smallmatrix} = F^{-1}(V_1)$ and hence F is fuzzy weakly δ-r_3-continuous.

We note that $I_\sigma(V_1,r) = V_1$ and $\mathbb{D}_\tau(\mu - F^{-1}(V_1))\begin{smallmatrix}[0.4,0.2]\\ [x,y]\end{smallmatrix}, r) = \bigwedge U_n \neq \mu - F^{-1}(V_1)$. Thus, F is not fuzzy weakly δ-r_4-continuous.

Counterexample 4: Let X = $\{x,y\}$, S = $\{s,t\}$. Define the fuzzy subsets $\mu\{\begin{smallmatrix}[0.8,0.6]\\ x,y\end{smallmatrix} \in I^X, \nu\begin{smallmatrix}[0.8,0.8]\\ [s,t]\end{smallmatrix} \in I^S$, $U_n\begin{smallmatrix}[0.4+\frac{1}{n+10},0.4+\frac{1}{n+10}]\\ [x,y]\end{smallmatrix}$, where n = 1,2, … and $V_1\begin{smallmatrix}[0.4,0.4]\\ [s,t]\end{smallmatrix}$. Let $\tau : \mathcal{J}_\mu \to I$ and $\sigma : \mathcal{J}_\nu \to I$ be defined by

$$\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu \\ 0.6, & U = U_n \forall n \quad \text{or} \quad \bigvee U_n, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\sigma(V) = \begin{cases} 1, & V = 0_S \text{ or } \nu \\ 0.5, & U = V_1, \\ 0, & \text{otherwise} \end{cases}$$

If $F : (\mu,\tau) \to (\nu,\sigma)$ is defined by

$F(x,s) = 0.8$, $F(x,t) = 0$, $F(y,s) = 0$, $F(y,t) = 0$,

Then F is one-to-one and $F(\mu)\,{}^{[0.8,0.6]}_{[s,t]} \neq \nu$. We fix $r = 0.5$. From $C_\sigma(V_1,r) = V_1 = I_\sigma(V_1,r) = V_1$, we have $\mathbb{D}_\sigma(V_1,r) = V_1$. Since $C_\tau(U_n,r) = U_n = I_\tau(U_n,r)$, $n = 1,2,\ldots$, we get that

$$F^{-1}(V_1)\,{}^{[0.4,0.4]}_{x,y} \leq I_\tau(U_n,r)$$

and

$$C_\tau(I_\tau(U_n,r),r) = U_n.$$

Therefore, $\mathbb{D}_\tau(F^{-1}(V_1),r)) = (\wedge U_n)\,{}^{[0.4,0.4]}_{[x,y]} = F^{-1}(V_1)$ and hence F is fuzzy weakly δ-r_3-continuous. From the observations, $I_\sigma(V_1,r) = V_1$ and $\mathbb{D}_\tau(\mu - F^{-1}(V_1)),r) = \wedge U_n \neq \mu - F^{-1}(V_1)$, we conclude that F is not fuzzy weakly δ-r_4-continuous.

The following counterexample shows that fuzzy weakly δ-r_4-continuous function is not a fuzzy weakly δ-continuous function.

Counterexample 5: Let X $=\{x,y\}, S = \{s,t\}$. Define $\mu\{{}^{[0.8,\,0.7]}_{[x,y]} \in I^X, \nu\,{}^{[0.8,0.7]}_{[s,t]} \in I^S$ and $V_1\,{}^{[0.4,0.3]}_{[s,t]} \in \mathcal{J}_\nu$

If $\tau : \mathcal{J}_\mu \to I$ and $\sigma : \mathcal{J}_\nu \to I$ are defined by

$$\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu \\ 0, & \text{otherwise} \end{cases}$$

and

$$\sigma(V) = \begin{cases} 1, & V = 0_S \text{ or } \nu \\ 0.5, & V = V_1, \\ 0, & \text{otherwise,} \end{cases}$$

then (μ,τ) and (ν,σ) are smooth fuzzy topological spaces. Let the fuzzy proper function $F : (\mu,\tau) \to (\nu,\sigma)$ be defined by

$F(x,s) = 0.8$, $F(x,t) = 0$, $F(y,s) = 0$, $F(y,t) = 0.7$.

Fix $r = 0.5$. If $I_\sigma(C_\sigma(V,r),r) = V$, then $V = 0_S$ or $V = \nu$ or $V = V_1$. But $C_\sigma\left((\nu - V_1)\,{}^{[0.4,0.4]}_{[s,t]}, r\right) \nleq V_1$ implies that $\mathbb{I}_\sigma(V_1,r) = 0_S$. Since $\mathbb{D}_\tau(\mu - F^{-1}(V), r) = \mu - F^{-1}(V)$, for every V with $\mathbb{I}_\sigma(V,r) = V$, we get that F is fuzzy weakly δ-r_4-continuous.

Next, we claim that F is not fuzzy weakly δ-continuous. Since $F\,(P_y^{0.45}) = P_t^{0.45} q V_1[\nu]$ and $I_\sigma(C_\sigma(V,r),r) = V_1$, V_1 is an R_σ^r-neighborhood of $P_t^{0.45}$. The only R_τ^r-neighborhood of $P_y^{0.45}$ is μ, for which we have $F(C_\tau(\mu,r)) = F(\mu) \nleq V_1$. Hence, our claim holds.

The proof of the following theorem is straightforward.

Theorem 6: *Let $r, q \in I_0$ and $F : (\mu,\tau) \to (\nu,\sigma)$.*

1. *If $r < q$ and if F is fuzzy weakly δ-r_1-continuous, then F is fuzzy weakly δ-$[r,q]_1$-continuous.*
2. *If $q < r$ and if $F : (\mu,\tau) \to (\nu,\sigma)$ is fuzzy weakly δ-$[r,q]_1$-continuous, then F is fuzzy weakly δ-r_1-continuous or F is fuzzy weakly δ-q_1-continuous.*
3. *If $r < q$ and if $F : (\mu,\tau) \to (\nu,\sigma)$ is fuzzy weakly δ-r_2-continuous, then F is fuzzy weakly δ-$[r,q]_2$-continuous.*
4. *If $q < r$ and if $F : (\mu,\tau) \to (\nu,\sigma)$ is fuzzy weakly δ-$[r,q]_2$-continuous, then F is fuzzy weakly δ-r_2-continuous and F is fuzzy weakly δ-q_2-continuous.*

Definition 12: *Let (μ,τ), (ν,σ) be smooth fuzzy topological spaces, $F : \mu, \to \nu$, be a fuzzy proper function and $r, q \in I_0$ be fixed. Then, F is called*

(1) fuzzy almost r_1-continuous or FAδ-r_1-C if
$F(C_\tau(A,r)) \leq \mathbb{D}_\sigma(F(A),r), \forall A \in \mathcal{J}_\mu$,

(2) fuzzy almost $[r,q]_1$-continuous or FAδ-$[r,q]_1$-C if
$F(C_\tau(A,r)) \leq \mathbb{D}_\sigma(F(A),q), \forall A \in \mathcal{J}_\mu$,

(3) fuzzy almost r_2-continuous or FAδ-r_2-C if
$C_\tau(F^{-1}(V),r) \leq F^{-1}(\mathbb{D}_\sigma(V,r)), \forall V \in \mathcal{J}_\nu$

(4) fuzzy almost $[r,q]_2$-continuous or FAδ-$[r,q]_2$-C if
$C_\tau(F^{-1}(V),r) \leq F^{-1}(\mathbb{D}_\sigma(V,q)), \forall V \in \mathcal{J}_\nu$,

(5) fuzzy almost r_3-continuous or FAδ-r_3-C if
$C_\tau(F^{-1}(V),r) \leq F^{-1}(V)$ *for each* $V \in \mathcal{J}_\nu$ *with* $V = \mathbb{D}_\sigma(V,r)$,

(6) fuzzy almost r_4-continuous or FAδ-r_4-C if
$C_\tau(\mu - F^{-1}(V),r) = \mu - F^{-1}(V)\ \forall V \in \mathcal{J}_\nu$ *with* $V = \mathbb{I}_\sigma(V,r)$.

Theorem 7: *Let $F : (\mu,\tau) \to (\nu,\sigma)$ be a one-to-one fuzzy proper function with $\nu = F(\mu)$. If F is fuzzy almost continuous, then F is fuzzy almost r_1-continuous.*

Since the proof of this theorem is similar to that of Theorem 4.7 in [11], we prefer to omit the details.

The statement of the above theorem is not true when F is not one-to-one $F(\mu) \neq \nu$. The following counterexamples justify our statement.

Counterexample 6: Let $X = \{x,y\}$, $S = \{s,t\}$. If define $\mu\{{}^{[0.7,\,0.5]}_{[x,y]} \in I^X, \nu\,{}^{[0.7,0]}_{[s,t]} \in I^S$, $U_1\,{}^{[0.3,0.3]}_{[x,y]} \in \mathcal{J}_\mu$ and $V_1\,{}^{[0.3,0]}_{[s,t]} \in \mathcal{J}_\nu$

We define smooth fuzzy topologies τ on μ and σ on ν by

$$\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu \\ 0.6, & U = U_1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\sigma(V) = \begin{cases} 1, & V = 0_S \text{ or } \nu \\ 0.5, & V = V_1, \\ 0, & \text{otherwise.} \end{cases}$$

Let the fuzzy proper function $F : (\mu, \tau) \to (\nu, \sigma)$ be defined by

$F(x,s) = 0.7$, $F(x,t) = 0$, $F(y,s) = 0.5$, $F(y,t) = 0$.

Then, F is not one-to-one and $F(\mu)\,{}^{[0.7,0]}_{[s,t]} = \nu$. We fix $r = 0.5$. For the R^r_σ-neighborhood V_1 of any $F(P^\eta_l)$, there exists U_1 as a Q^r_τ-neighborhood of P^η_l such that $F(U_1)\,{}^{[0.3,0]}_{[s,t]} = V_1$. For ν, we choose μ as a Q^r_τ-neighborhood P^η_l such that $F(\mu) = \nu$. Hence F is fuzzy almost continuous.

Since $P^{0.45}_y q U_1[\mu]$ and $P^{0.45}_y q \mu[\mu]$, U_1 and μ are the Q^r_τ-neighborhoods $P^{0.45}_y$. Clearly, we have $P^{0.45}_y \in C_\tau(A, r)$ and $F(P^{0.45}_y) = P^{0.45}_s = F\big(C_\tau(A, r)\big)$. Since,

$$V_1(s) + 0.45 > 0.7 = \nu(s) \text{ and } I_\sigma(C_\sigma(V_1, r), r) = C_\sigma((\nu - V_1)\,{}^{[0.4,0]}_{[s,t]}, r) = V_1,$$

we get that V_1 is an R^r_σ-neighborhood of $P^{0.45}_s$. Since $F(A)\,{}^{[0.3,0]}_{[s,t]}\,\bar{q}\,C_\sigma(V, r)[\nu]$, we have $P^{0.45}_s \notin \mathbb{D}_\sigma(F(A), r)$ and hence F is not fuzzy almost r_1-continuous.

Counterexample 7: Let X = $\{x, y\}$, $S = \{s, t\}$. Define the fuzzy subsets $\mu\,{}^{[0.7,\,0.6]}_{[x,y]} \in I^X$, $\nu\,{}^{[0.7,0.8]}_{[s,t]} \in I^S$, $U_1\,{}^{[0.3,0.3]}_{[x,y]} \in \mathcal{J}_\mu$, and $V_1\,{}^{[0.4,0.4]}_{[s,t]} \in \mathcal{J}_\nu$.
If $\tau : \mathcal{J}_\mu \to I$ and $\sigma : \mathcal{J}_\nu \to I$ are respectively, defined by

$$\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu \\ 0.6, & U = U_1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\sigma(V) = \begin{cases} 1, & V = 0_S \text{ or } \nu \\ 0.5, & V = V_1, \\ 0, & \text{otherwise.} \end{cases}$$

then (μ, τ) and (ν, σ) are smooth fuzzy topological spaces. If $F : (\mu, \tau) \to (\nu, \sigma)$ is defined by

$F(x,s) = 0.7$, $F(x,t) = 0$, $F(y,s) = 0$, $F(y,t) = 0.6$,

then as in the previous counterexample, we can verify that F is one-to-one, $F(\mu)\,{}^{[0.7,0.6]}_{[s,t]} \neq \nu$ and F is fuzzy almost continuous.

Next, we claim that $F(C_\tau(A, r)) \nleq \mathbb{D}_\sigma(F(A), r)$, for $A\,{}^{[0,0.4]}_{[x,y]} \in \mathcal{J}_\mu$. Since $P^{0.41}_y q U_1[\mu]$ and $P^{0.41}_y q \mu[\mu]$, we get that U_1 and μ are the Q^r_τ-neighborhoods $P^{0.41}_y$. We have, $U_1(y) + A(y) > 0.6 = \mu(y)$, $P^{0.41}_y \in C_\tau(A, r)$ and $F(P^{0.41}_y) = P^{0.41}_t = F\big(C_\tau(A, r)\big)$. Using $V_1(t) + 0.41 > 0.8 = \nu(t)$ and $I_\sigma(C_\sigma(V_1, r), r) = V_1$, we obtain that V_1 is an R^r_σ-neighborhood of $P^{0.41}_s$. However, $F(A)\,{}^{[0,0.4]}_{[s,t]}$ is not quasi-coincident with $C_\sigma(V, r)$ in ν. Therefore, F is not fuzzy almost r_1-continuous.

Theorem 8: *Let $F : (\mu, \tau) \to (\nu, \sigma)$ be a fuzzy proper function. If (a) F is fuzzy almost r_1-continuous, (b) F is fuzzy almost r_2-continuous, (c) F is fuzzy almost r_3-continuous, then (a) ⇒ (b) ⇒ (c).*

The proof of the theorem is straightforward.

Theorem 9: *Let $F : (\mu, \tau) \to (\nu, \sigma)$ be a one-to-one fuzzy proper function with $\nu = F(\mu)$. If F is fuzzy almost r_3-continuous, then F is almost r_4-continuous.*

Proof. If $V \in \mathcal{J}_\nu$ is such that $V = \mathbb{I}_\sigma(V, r)$, then $\nu - V = \nu - \mathbb{I}_\sigma(V, r) = \mathbb{D}_\sigma(\nu - V, r)$. Using hypothesis, we get $C_\tau(F^{-1}(\nu - V), r) = F^{-1}(\nu - V)$. Since F is one-to-one and $\nu = F(\mu)$, using Theorem 2, we have $F^{-1}(\nu - V) = \mu - F^{-1}(V)$. Therefore,

$$C_\tau(\mu - F^{-1}(V), r) = \mu - F^{-1}(V). \square$$

The statement of the above theorem is not true when F is not one-to-one or $F(\mu) \neq \nu$. The following counterexamples justify our statement.

Counterexample 8: Let $X = \{x, y\}$, $S = \{s, t\}$. We define, $\mu\,{}^{[0.8,0.6]}_{[x,y]} \in I^X$, $\nu\,{}^{[0.8,0]}_{[s,t]} \in I^S$, $U_1\,{}^{[0.4,0.2]}_{[x,y]} \in \mathcal{J}_\mu$, $V_1\,{}^{[0.4,0]}_{[s,t]} \in \mathcal{J}_\nu$.
We define $\tau : \mathcal{J}_\mu \to I$ and $\sigma : \mathcal{J}_\nu \to I$ by

$$\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu \\ 0.6, & U = U_1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\sigma(V) = \begin{cases} 1, & V = 0_S \text{ or } \nu, \\ 0.4, & U = V_1, \\ 0, & \text{otherwise.} \end{cases}$$

Let the fuzzy proper function $F : (\mu, \tau) \to (\nu, \sigma)$ be defined by

$F(x,s) = 0.8$, $F(x,t) = 0$, $F(y,s) = 0.6$, $F(y,t) = 0$.

We fix $r = 0.4$. Since $\mathbb{D}_\sigma(V_1, r) = V_1$ and $\mathbb{I}_\sigma(V_1, r) = V_1$, we obtain that $F^{-1}(V_1)\,{}^{[0.4,0.4]}_{[x,y]} = C_\tau(\mu - U_1, r)$. Hence, F is fuzzy almost r_3-continuous. But

$$C_\tau((\mu - F^{-1}(V_1))\,{}^{[0.4,\,0.2]}_{[x,\,y]}, r) = \mu - U_1 \neq U_1 = \mu - F^{-1}(V_1)$$

implies that F is not fuzzy almost r_4-continuous.

Counterexample 9: Let $X = \{x, y\}$, $S = \{s, t\}$, $\mu\,{}^{[0.8,0.6]}_{[x,y]} \in I^X, \nu\,{}^{[1,0.8]}_{[s,t]} \in I^S, U_1\,{}^{[0.3,0.2]}_{[x,y]}, V_1\,{}^{[0.5,0.4]}_{[s,t]}$.

If $\tau : \mathcal{J}_\mu \to I$ and $\sigma : \mathcal{J}_\nu \to I$ are respectively, defined by

$$\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu \\ 0.5, & U = U_1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\sigma(V) = \begin{cases} 1, & V = 0_S \text{ or } \nu, \\ 0.4, & U = V_1, \\ 0, & \text{otherwise.} \end{cases}$$

then (μ, τ) and (ν, σ) are smooth fuzzy topological spaces. If $F{:}(\mu,\tau) \to (\nu,\sigma)$ is defined by

$F(x,s) = 0.8$, $F(x,t) = 0$, $F(y,s) = 0$, $F(y,t) = 0.6$,

then $F(\mu)\,{}^{[0.8,0.6]}_{[s,t]} \neq \nu$. We fix $r = 0.4$. Since $C_\sigma(V_1, r) = V_1$ and $I_\sigma(V_1, r) = V_1$, we have $\mathbb{D}_\sigma(V_1, r) = V_1$. Using $F^{-1}(V_1)\,{}^{[0.5,0.4]}_{[x,y]} = \mu - U_1 = C_\tau(F^{-1}(V_1, r))$, we get that F is fuzzy almost r_3-continuous. From $\mathbb{I}_\sigma(V_1, r) = V_1$ and $C_\tau(\mu - F^{-1}(V_1), r) = \mu - U_1 \neq \mu - F^{-1}(V_1)$, we conclude that F is not fuzzy almost r_4-fuzzy continuous.

The following counterexample shows that F is fuzzy almost r_4-continuous but F is not fuzzy almost continuous.

Counterexample 10: Let $X = \{x, y\}$, $S = \{s, t\}$. Define $\mu\,{}^{[0.8,0.7]}_{[x,y]} \in I^X, \nu\,{}^{[0.8,0.7]}_{[s,t]} \in I^S$,

and $V_1\,{}^{[0.4,0.3]}_{[s,t]} \in \mathcal{J}_\nu$.

If $\tau : \mathcal{J}_\mu \to I$ and $\sigma : \mathcal{J}_\nu \to I$ are respectively, defined by

$$\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\sigma(V) = \begin{cases} 1, & V = 0_S \text{ or } \nu, \\ 0.5, & V = V_1, \\ 0, & \text{otherwise,} \end{cases}$$

then (μ, τ) and (ν, σ) are smooth fuzzy topological spaces. We define a fuzzy proper function $F : (\mu, \tau) \to (\nu, \sigma)$ by $F(x,s) = 0.8$, $F(x,t) = 0$, $F(y,s) = 0$, $F(y,t) = 0.7$. If $r = 0.5$, then $\mathbb{I}_\sigma(V_1, r) = 0_S$ and hence F is fuzzy almost r_4-continuous.

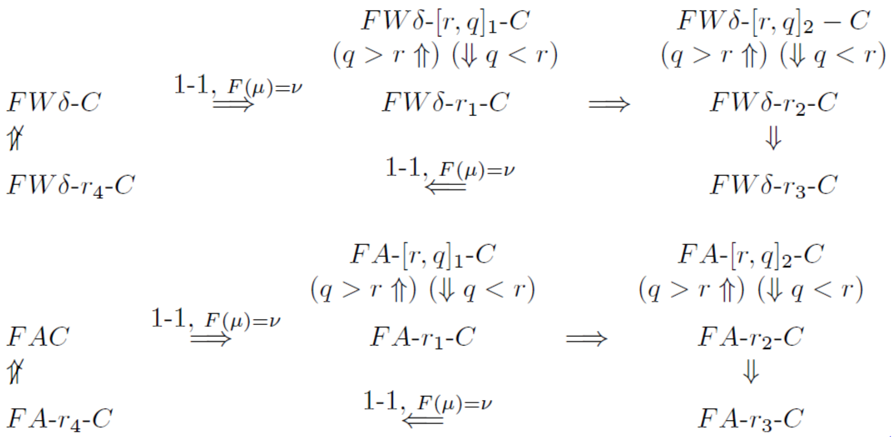
Clearly, V_1 *is* an R^r_σ-neighborhood of $F(P_y^{0.45}) = P_y^{0.45}$ and the only Q^r_τ-neighborhood of $P_y^{0.45}$ is μ. Since $F(\mu) \nleq V_1$, we get that F is not fuzzy almost continuous.

The proof of the following theorem is obvious.

Theorem 10: *Let $r, q \in I_0$ and $F : (\mu,\tau) \to (\nu,\sigma)$.*

1. *If $r < q$ and if F is fuzzy almost r_1-continuous, then F is fuzzy almost $[r,q]_1$-continuous.*
2. *If $q < r$ and if $F : (\mu,\tau) \to (\nu,\sigma)$ is fuzzy almost $[r,q]_1$-continuous, then F is fuzzy almost r_1-continuous and fuzzy almost q_1 continuous.*
3. *If $r < q$ and if $F : (\mu,\tau) \to (\nu,\sigma)$ is fuzzy almost r_2-continuous, then F is fuzzy almost $[r,q]_2$-continuous.*
4. *If $q < r$ and if $F : (\mu,\tau) \to (\nu,\sigma)$ is fuzzy almost $[r,q]_2$-continuous, then F is fuzzy almost r_2-continuous and F is fuzzy almost q_2-continuous.*

The results obtained in this section are summarized in the following implication diagram.



FUZZY q^r_α-PRE-CLOSURE AND FUZZY q^r_α-PRE-CONTINUOUS MAPS

Definition 13: *We say that U, V ∈ $\mathcal{J}_\mu$ are said to be α-quasi-coincident referred to μ [written as $Uq_\alpha V[\mu]$] if there exists x ∈ X such that U(x) + V(x) > μ (x) + α. If U is not α-quasi coincident with V, then we write $U\bar{q}_\alpha V[\mu]$.*

Definition 14: *A fuzzy set $U \in \mathcal{J}_\mu$ is called a fuzzy q^r_α-pre-neighborhood of α fuzzy point P_x^{λ} in μ if $P_x^{\lambda}\, q\alpha\, U[\mu]$ and U is r-preopen.*

Definition 15: *A fuzzy proper function* $F : \mu \to \nu$ *is said to be fuzzy* q_α^r*-pre-continuous if for every* q_α^r*-pre-neighborhood V of* $F(P_x^\lambda)$*, there exists a* q_α^r*-pre-neighborhood U of* P_x^λ *such that F(U)≤V.*

Definition 16: *Let* (μ,τ) *be a smooth fuzzy topological space and A∈* $\mathcal{J}_\mu$*. Then the fuzzy* q_α^r*-pre-closure* $PCl_\alpha(A, r)$ *of A is defined as follows:*

$\bigvee \{P_x^\lambda : U q_\alpha A[\mu]$
for every q_α^r*-pre-neighborhood U of* $P_x^\lambda\}$.

Theorem 11: *Let* (μ,τ) *be a smooth fuzzy topological space. For A, B ∈* $\mathcal{J}_\mu$*,* $r \in I_0$ *and* $\alpha \in I$*, this closure operator* PCl_α *satisfies the following properties:*

(1) $PCl_\alpha(0_X, r) = 0_X$,
(2) $A \le PCl_\alpha(A, r)$,
(3) $PCl_\alpha(A, r) \le PCl_\alpha(B, r)$ *if* $A \le B$,
(4) $PCl_\alpha(A, r) \vee PCl_\alpha(B, r) = PCl_\alpha(A \vee B, r)$,
(5) $PCl_\alpha(A \wedge B, r) \le PCl_\alpha(A, r) \wedge PCl_\alpha(A \vee B, r)$,
(6) $PCl_\alpha(PCl_\alpha(A, r), r) = PCl_\alpha(A, r)$.

Proof.

1. Clearly, $PCl_\alpha(0_X, r) = 0_X$.
2. Let $P_x^\lambda \in A$ and U be a q_α^r-pre-neighborhood of P_x^λ. Then, $A(x) \ge \lambda$ and $U(x) + \lambda$ $\mu(x) + \alpha$. Therefore, $A(x) + U(x) \ge \lambda + U(x) > \mu(x) + \alpha$. Thus, $Aq_\alpha U[\mu]$ and hence, $P_x^\lambda \in PCl_\alpha(A, r)$.
3. Let $A \le B$. Let $P_x^\lambda \in PCl_\alpha(A, r)$ and U be a q_α^r-pre-neighborhood of P_x^λ. Then, $Uq_\alpha A[\mu]$. Since $Uq_\alpha A[\mu]$ and $A \le B$, there exists $y \in X$ such that $U(y) + B(y) \ge U(y) + A(y) > \mu(y) + \alpha$, which implies that $P_x^\lambda \in PCl_\alpha(B, r)$. Thus, $PCl_\alpha(A, r) \le PCl_\alpha(B, r)$.
4. From (3), we get $PCl_\alpha(A, r) \vee PCl_\alpha(B, r) \le PCl_\alpha(A \vee B, r)$. If $P_x^\lambda \in PCl_\alpha(A \vee B, r)$ and U is a q_α^r-pre-neighborhood of P_x^λ, then $Uq_\alpha(A \vee B)[\mu]$. If $U\bar{q}_\alpha A[\mu]$ and $U\bar{q}_\alpha B[\mu]$, then $U + A \le \mu + \alpha$ and $U + B \le \mu + \alpha$. Hence, $Uq_\alpha(A \vee B)[\mu]$, which is a contradiction. Therefore, $PCl_\alpha(A, r) \vee PCl_\alpha(B, r) = PCl_\alpha(A \vee B, r)$.
5. By (3), we have $PCl_\alpha(A \wedge B, r) \le PCl_\alpha(A, r)$ and $PCl_\alpha(A \wedge B, r) \le PCl_\alpha(B, r)$. Thus, $PCl_\alpha(A \wedge B, r) \le PCl_\alpha(A, r) \wedge PCl_\alpha(B, r)$.
6. Again by using (3), we get $PCl_\alpha(A, r) \le PCl_\alpha(PCl_\alpha(A, r), r)$. If $P_x^\lambda \in PCl_\alpha(PCl_\alpha(A, r), r)$ and U is a q_α^r-pre-neighborhood of P_x^λ, then we have $Uq_\alpha PCl_\alpha(A, r)[\mu]$. Thererfore, we can find $s \in S$ such that $U(s) + PCl_\alpha(A, r)(s) \ge \mu(s) + \alpha$. If $\eta = PCl_\alpha(A, r)(s)$, then $P_s^\eta q_\alpha U[\mu]$ and $P_s^\eta \in PCl_\alpha(A, r)$. Therefore, $Uq_\alpha A[\mu]$ and hence $P_x^\lambda \in PCl_\alpha(A, r)$. □

The following counterexample shows that the equality does not hold in (5).

Counterexample 11: Let $X = \{x, y\}, \mu_{[x,y]}^{[0.6,0,5]} \in I^X$, $U_1{}_{[x,y]}^{[0.3,0.3]} \in \mathcal{J}_\mu$.

Define $\tau : \mathcal{J}_\mu \to I$ by $\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu \\ 0.6, & U = U_1 \\ 0, & \text{otherwise} \end{cases}$

We fix $\alpha = 0.1$, $r = 0.5$, $A_{[x,y]}^{[0.4,0.4]}$, and $B_{[x,y]}^{[0,0.5]}$.

Case 1. $0_X \ne U \le (\mu - U_1)_{[x,y]}^{[0.3,0.2]}$. In this case,

$$I_\tau(C_\tau(U, r), r) = I_\tau(\mu - U_1, r) = 0_X \not\ge U.$$

Case 2. $\mu \ne U \not\le \mu - U_1$. Here,

$$I_\tau(C_\tau(U, r), r) = I_\tau(\mu, r) = \mu \ge U.$$

If $U \in \mathcal{J}_\mu$ is such that $U(x) > 0.3$ or $U(y) > 0.2$, then U is r-pre-open. Then, the possible q_α^r-pre-neighborhoods of $P_y^{0.5}$ are $K_{[x,y]}^{[l,m]}$, where $l > 0.3$, $0.2 \ge m > 0.1$ and $J_{[x,y]}^{[l,m]}$, where $l \in [0,0.6]$, $m > 0.2$. The inequalities

$K(x) + A(x) > 0.3 + 0.4 = 0.7 = 0.6 + 0.1 = \mu(x) + \alpha$
$J(y) + A(y) > 0.2 + 0.4 = 0.6 = 0.5 + 0.1 = \mu(y) + \alpha$,

imply that $P_y^{0.5} \in PCl_\alpha(A, r)$. Clearly, $P_y^{0.5} \in PCl_\alpha(B, r)$ and hence $P_y^{0.5} \in PCl_\alpha(A, r) \wedge PCl_\alpha(B, r)$. We note that $K_{[x,y]}^{[0.35,0.15]}$ is a q_α^r-pre-neighborhood of $P_y^{0.5}$. But, $K(x) + (A \wedge B)(x) = 0.35 + 0 = 0.35 < 0.7 = \mu(x) + \alpha$ and $K(y) + (A \wedge B)(y) = 0.15 + 0.4 = 0.55 < 0.6 = \mu(y) + \alpha$ imply that $P_y^{0.5} \notin PCl_\alpha(A \wedge B, r)$.

Theorem 12: *Let* $F : (\mu, \tau) \to (\nu, \sigma)$ *be a one -to-one fuzzy proper function with* $\nu = F(\mu)$*. If F is fuzzy* q_α^r*-pre-continuous, then* $F(PCl_\alpha(A, r)) \le PCl_\alpha(F(A), r)$*, for every* $A \in \mathcal{J}_\mu$.

Proof. Suppose that $P_x^\lambda \in \mathcal{J}_\nu$ is such that $P_s^\lambda \notin PCl_\alpha(F(A), r)$. Since $F(\mu)(s) = \nu(s) \ge \lambda > 0$, there exists $x \in X$ such that $F(x,s) = \mu(x)$ and $F(P_x^\lambda) = P_s^\lambda$. On the other hand, there exists a q_α^r-pre-neighborhood V of $F(P_x^\lambda)$ such that $Vq_\alpha F(A)[\nu]$. Therefore, we get $V(s) + \lambda > \nu(s) + \alpha$ and $V + F(A) \le \nu + \alpha$. Since F is q_α^r-pre-continuous, there exists a q_α^r-pre-neighborhood U of P_x^λ such that $F(U) \le V$. Since F is one-to-one and $F(\mu) = \nu$, we get $U(x) + A(x) \le F(U)(s) + F(A)(s) \le V(s) + F(A)(s) \le \nu(s) + \alpha = \mu(x) + \alpha$. Therefore, $U\bar{q}_\alpha A[\mu]$ and $F(P_x^\lambda) \notin F(PCl_\alpha(A, r))$. Hence, $F(PCl_\alpha(A, r)) \le PCl_\alpha(F(A), r)$, for every $A \in \mathcal{J}_\mu$.

The statement of the above theorem fails to be true when F is not one-to-one and $F(\mu) \ne \nu$. The following counterexamples justify our statement.

Counterexample 12: Let $X = \{x,y\}$, $S = \{s,t\}$. Define $\mu\,^{[0.6,0.5]}_{[x,y]} \in I^X$, $\nu\,^{[0.6,0]}_{[s,t]} \in I^S$, $U_1\,^{[0.3,0.3]}_{[x,y]} \in \mathcal{J}_\mu$, and $V_1\,^{[0.4,0]}_{[s,t]} \in \mathcal{J}_\nu$.

If $\tau : \mathcal{J}_\mu \to I$ and $\sigma : \mathcal{J}_\nu \to I$ are respectively, defined by

$$\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu \\ 0.6, & U = U_1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\sigma(V) = \begin{cases} 1, & V = 0_S \text{ or } \nu, \\ 0.5, & V = V_1, \\ 0, & \text{otherwise}, \end{cases}$$

then (μ,τ) and (ν,σ) are smooth fuzzy topological spaces. If $F : (\mu,\tau) \to (\nu,\sigma)$ is defined by

$$F(x,s) = 0.6,\ F(x,t) = 0,\ F(y,s) = 0.5,\ F(y,t) = 0,$$

then F is not one-to-one and $F(\mu)^{[0.6,0]}_{[s,t]} = \nu$. Fix r = 0.5 and α = 0.1. First, we find all r-preopen sets in μ and ν. Clearly, 0_X, μ are r-preopen sets.

Case 1. $0_S \neq U \leq (\mu - U_1)\,^{[0.3,0.2]}_{[s,t]}$. In this case, $I_\tau(C_\tau(U,r),r) = I_\tau(\mu - U_1,r) = 0 \ngeq U.$ Hence, , each U is not r-preopen.

Case 2. $\mu \neq U \nleq (\mu - U_1)$. Here, $I_\tau(C_\tau(U,r),r) = I_\tau(\mu,r) = \mu \geq U$. Hence, $U\,^{[p,q]}_{[x,y]}$ is r-preopen, whenever $p > 0.3$ or $q > 0.2$. Next we find all r-preopen sets in ν. Clearly, 0_S, ν are r-preopen sets.

Case (i). $0_S \neq V \leq (\nu - V_1)\,^{[0.2,0]}_{[s,t]}$. In this case, $I_\sigma(C_\sigma(V,r),r) = I_\sigma(\nu - V_1,r) = 0 \ngeq V$. Hence, each V is not r-preopen.

Case (ii). $\nu \neq V \nleq (\nu - V_1)$. Here, $I_\sigma(C_\sigma(V,r),r) = I_\sigma(\nu,r) = \nu \geq V$. Hence, $V\,^{[p,q]}_{[s,0]}$ is r-preopen, where $p > 0.2$. We claim that F is q^r_α-pre-continuous. Clearly, ν is a q^r_α-pre-neighborhood of both $F(P^\lambda_x)$ and $F(P^\eta_y)$. For ν, we choose μ as a required q^r_α-pre-neighborhood of both P^λ_x, P^η_y such that $F(\mu) = \nu$. Let $V\,^{[l,0]}_{[s,t]}$ be a q^r_α-pre-neighborhood of $F(P^\lambda_x) = P^\lambda_s$. Since V is r-preopen, we have $l > 0.2$.

Case (a). $0.3 \geq l > 0.2$. If we choose $U\,^{[l,m]}_{[x,y]}$ with $l \geq m > 0.2$, then $U(x) + \lambda = l + \lambda = V(s) + \lambda > \nu(s) + \alpha = 0.6 + 0.1 = \mu(x) + \alpha$. Since $U(y) = m > 0.2$, U is r-preopen. Therefore, U is a q^r_α-pre-neighborhood of P^λ_x such that $F(U)^{[l,0]}_{[s,t]} = V$.

Case (b). $l > 0.3$. In this case, we choose $U^{[l,0]}_{[x,y]}$ as a required q^r_α-pre-neighborhood of P^λ_x such that $F(U)^{[l,0]}_{[s,t]} = V$.

Let $V\,^{[l,0]}_{[s,t]}$ be a q^r_α-pre-neighborhood of $F(P^\eta_y) = P^\eta_s$. Since V is r-preopen, we have $l > 0.2$. Clearly, $U^{[0,l]}_{[x,y]}$ is r-preopen and $U(y) + \eta = V(s) + \eta > 0.7 > 0.5 + 0.1 = \mu(y) + \alpha$ and hence F is q^r_α-pre-continuous. Now, we claim that $F(PCl_\alpha(A, r)) \nleq PCl_\alpha(A, r)$ for $A^{[0.4,0.4]}_{[x,y]}$. The possible r-preopen sets of $P^{0.5}_x$ are $K^{[l,m]}_{[x,y]}$, where $0.3 \geq l > 0.2$ and $m > 0.2$ and $J^{[l,m]}_{[x,y]}$, where $l > 0.3$ and $m \in [0, 0.5]$. Since $K(y) + A(y) > \mu(y) + \alpha$ and $J(x) + A(x) > \mu(x) + \alpha$, we get that $F(P^{0.5}_x) \in F(PCl_\alpha(A\ ,r))$. Clearly, $V^{[0.21,0]}_{[s,t]}$ is r-preopen and $V(s) + 0.5 > \nu(s) + \alpha$. Since $F(A)^{[0.4,0]}_{[s,t]}$, we have $V(s) + F(A)(s) < \nu(s) + \alpha$ and $V(t) + F(A)(t) = 0 < \nu(t) + \alpha$. Therefore, V is a q^r_α-pre-neighborhood of $F(P^{0.5}_x)$ and $V\bar{q}_\alpha F(A)[\nu]$. Thus, $F(P^{0.5}_x) \notin PCl_\alpha(F(A), r)$.

Counterexample 13: Let $X = \{x,y\}$, $S = \{s,t\}$. Define $\mu\,^{[0.7,0.6]}_{[x,y]} \in I^X$, $\nu\,^{[0.8,0.8]}_{[s,t]} \in I^S$, $U_1\,^{[0.4,0.3]}_{[x,y]} \in \mathcal{J}_\mu$, and $V_1\,^{[0.5,0.5]}_{[s,t]} \in \mathcal{J}_\nu$.

Let $\tau : \mathcal{J}_\mu \to I$ and $\sigma : \mathcal{J}_\nu \to I$ be defined by

$$\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu \\ 0.6, & U = U_1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\sigma(V) = \begin{cases} 1, & V = 0_S \text{ or } \nu, \\ 0.5, & V = V_1, \\ 0, & \text{otherwise}, \end{cases}$$

If $F:(\mu,\tau) \to (\nu,\sigma)$ is defined by

$$F(x,s) = 0.7,\ F(x,t) = 0,\ F(y,s) = 0,\ F(y,t) = 0.6,$$

then F is one-to-one and $F(\mu)\,^{[0.7,0.6]}_{[s,t]} = \nu$. We fix r = 0.5 and α = 0.2. First, we find all r-preopen sets in μ. Clearly, 0_X and μ are r-preopen sets.

Case 1. $0_S \neq U \leq (\mu - U_1)\,^{[0.3,0.3]}_{[s,t]}$. In this case, $I_\tau(C_\tau(U,r),r) = I_\tau(\mu - U_1,r) = 0 \ngeq U$. Hence, each U is not r-preopen.

Case 2. $\mu \neq U \nleq (\mu - U_1)$. Here, $I_\tau(C_\tau(U,r),r) = I_\tau(\mu,r) = \mu \geq U$. Hence, each $U\,^{[p,q]}_{[x,y]}$, is r-preopen, whenever $p > 0.3$ or $q > 0.3$.

Similarly, we can verify that 0_S, ν and each $V\,^{[p,q]}_{[s,0]}$ is r-preopen, where $p > 0.3$, $q > 0.3$. We claim that F is q_α^r-pre-continuous. Let $V_{[s,t]}^{[l,m]}$ be a q_α^r-pre-neighborhood of $F(P_x^\lambda) = P_s^\lambda$. Since $V_{[s,t]}^{[l,m]}$ is r-preopen, we have $l > 0.3$ or $m > 0.3$. If we choose $U_{[x,y]}^{[p,q]}$ with $p = l$ and $q = m$, then $U(x) + \lambda = l + \lambda = V(s) + \lambda > \nu(s) + \alpha = 1 > 0.9 = \mu(x) + \alpha$. Since $U(x) > 0.3$ or $U(y) > 0.3$, U is r-preopen. Therefore, U is a q_α^r-pre-neighborhood of P_x^λ such that $F(U)_{[s,t]}^{[l,m]} \leq V$.

Let $V_{[s,t]}^{[l,m]}$ be a q_α^r-pre-neighborhood of $F(P_y^\eta) = P_t^\eta$. Since $V_{[s,t]}^{[l,m]}$ is r-preopen, $l > 0.3$ or $m > 0.3$, we choose $U_{[x,y]}^{[p,q]}$ with $p = l$ and $q = m$ so that U is a q_α^r-pre-neighborhood of P_y^η such that $F(U)_{[s,t]}^{[l,m]} \leq V$.

Next, we claim that $F(PCl_\alpha(A, r)) \nleq PCl_\alpha(A, r)$, for $A_{[x,y]}^{[0.6,0.5]}$. Consider $P_y^{0.6} \in \mu$. The possible r-preopen sets of $P_y^{0.6}$ are $K_{[x,y]}^{[l,m]}$, where $l \in [0, 0.7]$ and $m > 0.3$ and $J_{[x,y]}^{[l,m]}$, where $l > 0.3$ and $m \in [0, 0.6]$. From the inequalities

$$K(y) + A(y) > 0.3 + 0.5 > \mu(y) + \alpha$$
$$J(x) + A(x) > 0.3 + 0.6 = 0.9 = \mu(x) + \alpha,$$

we get that $P_y^{0.6} \in F(PCl_\alpha(A, r))$. Clearly, we have $V_{[s,t]}^{[0,0.41]}$ is r-preopen and $V(t) + 0.6 > \nu(t) + \alpha$. Since $F(A)_{[s,t]}^{[0.6,0.5]}$, we have $V(s) + F(A)(s) = 0 + 0.6 < \nu(s) + \alpha$ and $V(t) + F(A)(t) < \nu(t) + \alpha$. Thus, $P_y^{0.6} \notin PCl_\alpha(F(A), r)$.

The following counterexample shows that the converse of Theorem 12 is not true.

Counterexample 14. Let $X = \{x,y\}$, $S = \{s,t\}$. Define $\mu_{[x,y]}^{[0.8,0.6]} \in I^X$, $\nu_{[s,t]}^{[0.8,0.6]} \in I^S$, $U_1{}_{[x,y]}^{[0.5,0.3]} \in \mathcal{J}_\mu$, and $V_1{}_{[s,t]}^{[0.4,0.3]} \in \mathcal{J}_\nu$.

If $\tau : \mathcal{J}_\mu \to I$ and $\sigma : \mathcal{J}_\nu \to I$ are respectively, defined by

$$\tau(U) = \begin{cases} 1, & U = 0_X \text{ or } \mu \\ 0.6, & U = U_1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\sigma(V) = \begin{cases} 1, & V = 0_S \text{ or } \nu, \\ 0.5, & V = V_1, \\ 0, & \text{otherwise}, \end{cases}$$

Let the fuzzy proper function $F : (\mu, \tau) \to (\nu, \sigma)$ be defined by

$$F(x,s) = 0.8,\ F(x,t) = 0,\ F(y,s) = 0,\ F(y,t) = 0.6.$$

We note that F is one-to-one and $F(\mu)_{[s,t]}^{[0.8,0.6]} = \nu$. Fix $r = 0.5$ and $\alpha = 0.2$. First, we find all r-preopen sets in μ. Clearly, 0_X and μ are r-preopen sets.

Case 1. $0_S \neq U \leq (\mu - U_1)_{[s,t]}^{[0.3,0.3]}$. In this case, $I_\tau(C_\tau(U,r),r) = I_\tau(\mu - U_1, r) = 0 \ngeq U$. Hence, each U is not r-preopen.

Case 2. $\mu \neq U \nleq (\mu - U_1)$. Now, $I_\tau(C_\tau(U,r),r) = I_\tau(\mu, r) = \mu \geq U$. Hence, if $p > 0.3$ or $q > 0.3$, then $U_{[x,y]}^{[p,q]}$ is an r-preopen fuzzy set.

Next, we find all r-preopen sets in ν. Clearly, 0_S and ν are r-preopen sets.

Case (i). $0_S \neq V \leq (\nu - V_1)_{[s,t]}^{[0.4,0.3]}$. In this case, $I_\sigma(C_\sigma(V,r),r) = I_\sigma(\nu - V_1, r) = V_1 = \nu - V_1 \geq V$. Hence, each V is $r-$preopen.

Case (ii). $\nu \neq V \nleq (\nu - V_1)$. Here, $I_\sigma(C_\sigma(V,r),r) = I_\sigma(\nu, r) = \nu \geq V$. Hence, each $V_{[s,t]}^{[p,q]}$ is r-preopen.

Clearly, $V_{[s,t]}^{[0,0.3]}$ an is r-preopen fuzzy subset in $\mathcal{J}_\mu$ and from the inequality $V(t) + 0.55 = 0.3 + 0.55 > 0.8 = \nu(t) + \alpha$, we have V is a q_α^r-pre-neighborhood of $F(P_y^{0.55}) = P_t^{0.55}$. Using that the q_α^r-pre-neighborhoods of $P_y^{0.55}$ are $K_{[x,y]}^{[l,m]}$, where $l \in [0, 1]$ and $m > 0.3$ and $J_{[x,y]}^{[l,m]}$, where $p > 0.3$ and $0.3 \geq q > 0.25$ and $F(K)_{[x,y]}^{[l,m]} \nleq V$, $F(J)_{[x,y]}^{[l,m]} \nleq V$, we conclude that F is not q_α^r-pre-continuous. We claim that $F(PCl_\alpha(A, r)) \leq PCl_\alpha(F(A), r)$, for every $A \in \mathcal{J}_\mu$. Let $A_{[x,y]}^{[l,m]}$.

Case (a). $l \geq 0.7$ or $m \geq 0.5$. In this case, every r-preopen set $U_{[x,y]}^{[p,q]}$ is α-quasi coincident with A, where $p > 0.3$ or $q > 0.3$. Therefore, $F(PCl_\alpha(A, r) = F(\mu)$. Clearly, $F(A)(s) = l \geq 0.7$ or $F(A)(t) = m \geq 0.5$. Hence, $PCl_\alpha(F(A), r) = \nu = F(\mu) = F(PCl_\alpha(A, r))$.

Case (b). $l < 0.7$ and $m < 0.5$. Clearly, $A \leq PCl_\alpha(A, r)$. Suppose that $\lambda > A(x) = l$. We can choose a q_α^r-pre-neighborhood $U_{[x,y]}^{[p,q]}$ of P_x^λ, where $\mu(x) - l > p > \mu(x) - \lambda$ and $q > 0.3$. Therefore, U is α_α^r-pre-neighborhood of P_x^λ but $U \bar{q}_\alpha A[\mu]$. For any $\eta > A(y) = m$, we can choose a q_α^r-pre-neighborhood $W_{[x,y]}^{[p,q]}$ of P_y^η where $\mu(y) - m > q > \mu(y) - \eta$ and $p > 0.3$. Therefore, W is α_α^r-pre-neighborhood of P_y^η but $W \bar{q}_\alpha A[\mu]$. Thus, $PCl_\alpha(A, r) = A$.

Hence, $F(PCl_\alpha(A, r)) = F(A) \leq PCl_\alpha(F(A), r)$.

CONCLUSION

Using different notions of fuzzy closure operators, we have introduced various notions of weaker forms of continuities such as fuzzy weakly δ-continuity, fuzzy weakly δ-r_1-continuity, fuzzy weakly δ-r_2-continuity, fuzzy weakly δ-r_3-continuity, etc., and inter-relations among them are obtained completely. Further, we have introduced new notion of quasi coincidence namely α-quasi coincidence and then a fuzzy closure operator PCl_α is introduced. Using this fuzzy closure operator, q_α^r -pre-continuous fuzzy proper function is introduced and all properties of this function are obtained.

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НЕКОИ ПОСЛАБИ ФОРМИ НА МАЗНИ ФАЗИ НЕПРЕКИНАТИ ПРЕСЛИКУВАЊА

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Во овој труд, воведуваме неколку поими за непрекинати фази прави пресликувања, со користење на постоечките поими за операторите фази затворач и фази внатрешност, како што се R_τ^r-затворач, R_τ^r-внатрешнос итн, и ги изнесуваме сите можни врски помеѓу тие типови на непрекинатости. Понатаму ги воведуваме концептите за α-квази-коинциденија, q_α^r-пре-околина, q_α^r-пре-затворач и q_α^r-пре-непрекинати пресликувања во мазни фази тополошки простори и ги испитуваме еквивалентните услови за q_α^r-пре-непрекинатост.

Клучни зборови: Фази прави преслиувања; мазна фази топологија; мазна фази непрекинатост; фази затворач; фази внатрешност