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Educational paper

THE ROLE OF VISUALIZATION IN UNDERGRADUATE MATHEMATICS

Mile Krajčevski

University of South Florida, United States of America
mile@usf.edu

We emphasize the importance of visualization in undergraduate mathematics courses and suggest drawing-to-learn intervention that will help students solidify concept images of mathematical objects through drawing activity.

Key words: visualization, visualization object, undergraduate mathematics

INTRODUCTION

Visualization as a process of reflection upon pictures, images or diagrams on a blackboard, paper or other technical tools, is a subject of intense research in the last decade. Deductive and analytical nature of mathematical reasoning seems detached from the influence of visual images designed to illustrate the connection between the given data and the unknown in a particular mathematical problem. In recent years we are witnessing a tremendous development of computer graphics and educational software. Their ubiquitous penetration into educational practice may lead some to believe that nonparticipative exposure of the learner to predetermined images and spatial representations of mathematical objects on a computer screen can be an alternative to the process of active creation of visual representation of mathematical objects by hand, and opportunity to manipulate them in this creative process. The process of sketching abstract mathematical objects involves powerful hand-mind coordination which results in concretization of these objects as creations of our own hands, subject to easy manipulation or transformation. Although we are witnessing an increasing demand for profiles in science, technology, engineering and mathematics (STEM), more than 40% of STEM majors in US universities switch to non-STEM majors before graduation [8]. Mathematics education researchers' attempt to analyse this problem, among other factors, emphasize the weakening standards of high school mathematical curriculum and not enough attention has been paid to different modes of presenting mathematical content.

Inspiration for this essay is the work of Quillin and Thomas [15] in which they create a framework for drawing-to-learn approach to reasoning in biology classroom. Our research in using visualisation in teaching of Calculus III, Vector Calculus and Linear Algebra courses taught at university level shares many of their findings and suggestions. We have demonstrated that the reluctance to use visualization as a tool in problem-solving strategies is not correlated with students' ability to sketch, but rather to the predominance of the analytic way of presenting mathematical statements in the school curriculum. Hesitancy towards adequate use of visual arguments in the process of justification (proof), adds to the perceived bias towards visualizing mathematical statements, and deprives students of a powerful cognitive tool. Freehand drawing of mathematical objects (lines, planes, spheres, etc.) has all the elements of modelling and creates an opportunity for a learner to manipulate an abstract mathematical object and serves as a cognitive tool in the learning process.

WHAT IS VISUALIZATION

There is no unified definition of the term "visualization" in mathematics education literature. In their influential work on visualization in mathematics, reading and science education Phillips, Norris and Macnab give 28 explicit definitions of visualisation and related terms, in education literature from 1974 to 2010 [12]. In the educational literature one can find multiple usages for the same term, sometimes even contradicting each other. In what follows, we adapt their definition of "visualization object", but we use the term "visualization" in drawing-to-learn activity as a verb.

Visualization objects are physical objects that are viewed and interpreted by a person for the purpose of understanding something other than the object itself. These objects can be pictures, 3D representations, schematic representations, animations, etc. Other sensory data such as sound can be integral parts of these objects and the objects may appear on many media such as paper, computer screens and slides.

Bishop [2] was the first one to note the important distinction between use of the term "visualization" as a noun and as a verb. The noun "directs our attention to the product, the object, the 'what' of visualization, the visual images. The verb of visualization, on the other hand, make us attend to the process, the activity, the skill, the 'how' of visualizing". He defines "visual processing ability" as "ability that involves visualization and the translation of abstract relationships and non-figural information into visual terms. It also includes manipulation and transformation of visual representations and visual imagery. It is an ability of process and does not relate to the form of the stimulus material presented." [2]

Advancement of electronic devices and tools for drawing and computer-generated animations necessitated modification of this definition, resulting in the above-mentioned definition in [12].

Like in the case of visualization, there is no unified approach or definition of drawing in drawing-to-learn notion. One can adopt an inclusive definition of drawing given in [15] for the purposes of drawing mathematical objects or mathematical notions, broadly defined as:

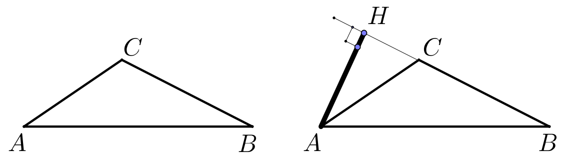
a learner-generated external visual representation depicting any type of mathematical object, whether structure, relationship, or process, created in static two dimensions on any medium.

We should note that creating an external model of a mathematical object requires not only mental processes, but also coordination of hand movements created by following some kinematic and kinetic parameters that will result in intended action. The point of mentioning this is that drawings presented by an experienced instructor can be intimidating for an unexperienced learner, thereby students should be constantly encouraged and reminded that artistic attributes of the visualized object in most cases are not a prerequisite for its successful use in the cognitive process.

Although in our teaching practice we are mostly focused on constructing visualization objects, either on a whiteboard during enacted lesson, on paper, or electronically on a computer screen, we should mention two other distinctive attributes of visualization as a process.

Introspective visualization refers to mental objects that a person/learner makes in building their *concept image*. The notion of a concept image and *concept definition* are two useful ways of understanding a mathematical concept. These were created by Tall and Vinner [18] and often visualization is discussed in the framework suggested in [18]. They define the notion of a concept image "…to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures". A concept definition on the other side, is similar to the notion of definition in mathematics, with the distinction of being personal to an individual. According to [18], "… a personal concept definition can differ from a formal concept definition, the latter being a concept definition which is accepted by the mathematical community at large".

Interpretive Visualization is an act of making meaning from a visualization object or an introspective visualization by interpreting information from the objects or introspections and by cognitively placing the interpretation within the person's existing network of beliefs, experiences, and understanding [12]. Our pedagogical practice shows that many high school and collegiate geometry students do not make the distinction between a mathematical object (notion) and their physical realization in the form of a visualization object or picture. Just as an illustration, if *AH* is the altitude from vertex *A* in the triangle *ABC* where we assume that the angle at the vertex *C* is an obtuse angle, for majority of geometry students, the altitude *AH* will not exist as a mathematical object, or it will not be introspectively visualized, unless drawn on the paper or a whiteboard.



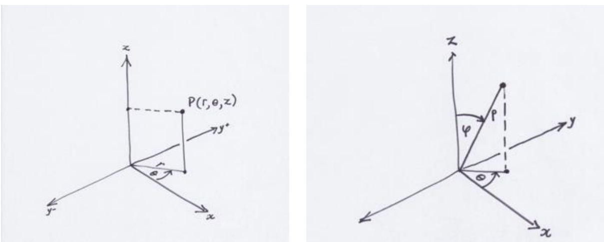
Introducing the altitude *AH* as an auxiliary element in the visual representation of the triangle *ABC*, will provide valuable insight on how to apply the basic formula for the area of a triangle if we take side *BC* to be a base of the triangle [13] (see p. 47).

WHAT CONSTITUTES A GOOD VISUAL INTERPRETATION?

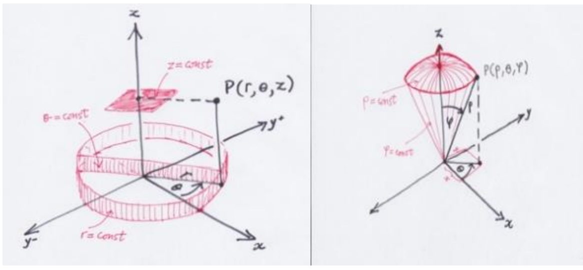
Research has demonstrated that for the vast majority of scientists and mathematicians' visualization plays central role in their cognitive processes. Comprehensive theory of visual images in mathematical education is still lacking, and in absence of such theory, we rely more on intuition and scattered evidence on the use of images in the learning process or usefulness of drawing-to-learn activities embodied in the classroom practice.

In what follows, we will outline number of interventions that will help instructors create an environment conductive to students' drawing-to-learn activities in the classroom. Our experience and research have been conducted with students in Calculus and Linear Algebra classes. On few occasions we've worked with College of Education's prospective teachers in the Geometry Connection course. More information about research methods and findings can be found in [3,5,10,11].

Typical Calculus III material is especially suitable for drawing-to-learn approach. The example that follows was from enacted lesson to 45 engineering students and more details can be found in [10]. We note that students had previous experience with drawing 3D coordinate system and sets of points whose coordinates satisfy certain (simple, mostly linear) algebraic equation(s). We distinguished three categories of images presented during enacted lesson: primary image, secondary image, and second-layer image. We define *primary image* as an image on which the derivation of the analytical portion of the presentation rests, also related to a justification (visual proof) of subsequent proposition. In the following figure we provide two examples of primary images that were given in the lecture about triple integrals in spherical coordinate system.

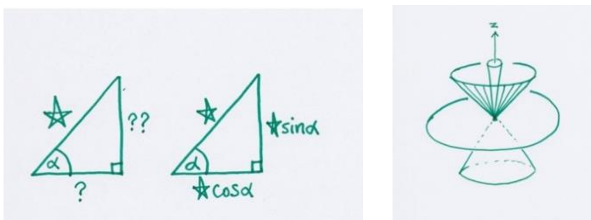


Primary image may also play an essential role in the explanation of a new mathematical concept. Usually, this image will stay on the board during an enacted lesson for a substantial amount of time (compared with the length of the class period). A *second-layer image* is an image that will be superimposed on a primary image later in the exposition, bringing new aspects of the presented notion, or illustrating a portion of the proof of a proposition. Most of the time, during the enacted lesson, students are inclined to sketch a completely new illustration rather than revisiting a primary image and superimposing on it a new one.



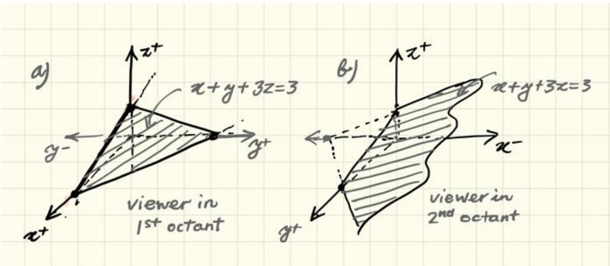
Illustrative elements that have been superimposed on the primary images are indicated as shaded areas in red. Two remarks are in place for the second-layer images. The first one is that we follow Inglis and Mejia-Ramos' suggestion that students are more inclined to accept figures as evidence for a claim if it is accompanied by descriptive text explaining the claim. In the enacted lesson, the second layer imagery has been used also to reinforce the previous notions (cylindrical coordinate system) and emphasize connections between the coordinate systems. Students were invited to derive analytical/algebraic expressions for (portions of the) surfaces shown and convince themselves about justification of introducing these new coordinate systems. The second note is about the use of color when presenting images of mathematical objects. Research shows that excessive use of color can be impeding to the cognitive process and could be experienced as a distraction to the learner.

Our third category of images are so called *secondary images*. These are images that have been used to clarify a particular argument related to the primary image, illustrate a particular point in the analytical portion of the argument, or used as a review of a specific notion used in the exposition. Usually, these images will stay a short time on the board, serving its purpose and not interfering with the primary image. We illustrate this category in the following figure.



The left picture on the above figure helps students recall the definition of sine and cosine function but in an unusual way. Usually, these functions are given as ratios of a length of a leg with length of the hypothenuse in a right triangle. Our practice shows unusual persistence of this high school concept image and prevents students to see different forms of this analytical formula. The right picture illustrates (parts of) four graphs of the equation $\varphi = c$ (c is a constant) in the spherical coordinate system, for four different choices of c all between 0 and π. Initially, after showing the cone with c close to zero, students are invited to sketch a cone with their specific choice of c.

One can notice that on previous figures, mathematical objects are presented in their "typical" position. In [5] we have examined the diversity of imagery of the same mathematical object (triangle, parallelogram and trapezoid) in high school geometry books used in majority of schools in Florida. Our definition of a typical images of a particular mathematical object is "…*a visual representation of that object that is drown in the majority of instances with no content-based reason"*. In our view on the 3D coordinate system, the viewer is in the first octant and this is the typical representation in calculus textbooks. The role of the instructor is to point this anomaly and to invite students to represent the same mathematical object in 3D but from different standpoint of the observer.



We have illustrated the need to draw in calculus classes, especially when working with undergraduate STEM majors, but similar arguments can be made that will advocate the use of visual arguments in mathematics classes, for much needed scaffolding when constructing proofs of a given propositions. To be clear, we are not advocating acceptance of visual argument and pictures, as proofs in mathematics. We seek interventions that will help student in establishing well organized and coherent library of concept images, as a necessary tool in the practice of proving mathematical theorems.

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УЛОГАТА НА ВИЗУАЛИЗАЦИЈАТА ВО НАСТАВАТА ПО МАТЕМАТИКА ВО ВИСОКОТО ОБРАЗОВАНИЕ

Миле Крајчевски

Универзитет на Јужна Флорида, САД

Овој краток есеј ја обработува темата на визуализација на математичките објекти и поими во методиката на наставата по математика во високото образование. Покрај дефинициите на визуализација и поимот цртам-да-научам, низ конкретни примери се илустрира важноста на овој начин на презентација на математичка содржина особено во техничките науки.

Клучни зборови: визуализација, визуелен објект, високо образование

Има многу да се раскажува за влијанието на професор Чупона не само на мојот пат во математиката, туку и на патот на една цела генерација од македонски математичари и педагози. Тој имаше дарба да почувствува кои математички прашања би биле интересни за соговорникот и не ги наметнуваше неговите погледи како нешто што треба да се следи. Но пред се, имаше визија како треба да се унапредува и развива математичката мисла во Македонија и умешно им сугерираше на студентите каде нивниот потенцијал најдобро ќе дојде до израз. Беше свесен за потребата на дидактичките и методичките истражување во математиката и ми сугерираше да работам на проблемите поврзани со наставата по математика во високото образование.