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ON FREE GROUPOIDS WITH $(xy)^n = x^n y^n$

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We investigate free objects in the variety of groupoids which satisfy the identity $(xy)^n = x^n y^n$**.** Under certain condition for the groupoid power x^n, i.e. for simple groupoid powers, a canonical description for free groupoids in such varieties is given and they are characterized by the injective groupoids in these varieties.

Key words: variety of groupoids, free groupoid, groupoid powers

INTRODUCTION

In the papers [3,5,6,8,9], Čupona and coauthors investigated free objects in varieties of groupoids satisfying some identities among groupoid powers. Free objects in the variety of groupoids satisfying the law $(xy)^2 = x^2y^2$ are investigated in [4]. Almost 20 years ago, together with Professor Čupona, we obtained a canonical description of free objects in the variety of groupoids satisfying the identity $(xy)^n = x^n y^n$ for some groupoid powers x^n. This result was not published, and the question of finding a canonical description of free objects for an arbitrary groupoid power x^n is still open. In this paper we present a slight improvement of the above mentioned, canonical description.

First, we state some necessary preliminaries.

Let $\boldsymbol{G} = (G,\cdot)$ be a *groupoid*, i.e. an algebra with a binary operation $(x, y) \to xy$ on G. If $a = bc$ for $a, b, c \in G$, we say that b, c are *divisors* of a in $\boldsymbol{G}$. A sequence $a_1, a_2, \ldots$ of elements of G is said to be a *divisor chain* in $\boldsymbol{G}$ if a_{i+1} is a divisor of a_i. We say that $a \in G$ is a *prime* in $\boldsymbol{G}$ if the set of divisors of a in $\boldsymbol{G}$ is empty. A groupoid $\boldsymbol{G} = (G,\cdot)$ is said to be *injective* if $xy = uv$ implies $(x, y) = (u, v)$, for any $x, y, u, v \in G$. By a "free groupoid" we mean "free groupoid in the variety of groupoids" (i.e. an "absolutely free groupoid").

The following characterization of free groupoids is well known (see for example [1], I.1.)

Theorem. 1.1 *A groupoid $\boldsymbol{F} = (F,\cdot)$ is free if and only if (iff) it satisfies the following conditions.*

(1) *Every divisor chain in $\boldsymbol{F}$ is finite.*
(2) *$\boldsymbol{F}$ is injective.*

Then the set B of primes in $\boldsymbol{F}$ is nonempty and it is the unique basis of $\boldsymbol{F}$. ■

Throughout the paper, a free groupoid with basis B will be denoted by $\boldsymbol{F}$ or $\boldsymbol{F}(B)$. For any $v \in F$, we define the *length* $|v|$ and the *set* $P(v)$ *of parts of* v by:

$$|b| = 1, \quad |tu| = |t| + |u|$$
$$P(b) = \{b\},\ P(tu) = \{tu\} \cup P(t) \cup P(u)$$

for every $b \in B,\ t, u \in F$.

GROUPOID POWERS

We recall some definitions, notions and statements from [7].

Let $\boldsymbol{E} = (E,\cdot)$ be a free groupoid with one-element basis $\{e\}$. The elements of E will be denoted by $f, g, h, \ldots$ and called *groupoid powers.*

If $\boldsymbol{G} = (G,\cdot)$ is a groupoid, then each $f \in E$ induces a transformation $f^{\boldsymbol{G}}$ of G (called the *interpretation* of f in $\boldsymbol{G}$) defined by:

$$f^{\boldsymbol{G}}(x) = \varphi_x(f)$$

where $\varphi_x: E \to G$ is the unique homomorphism from $\boldsymbol{E}$ to $\boldsymbol{G}$ such that $\varphi_x(e) = x$. In other words

$$e^{\boldsymbol{G}}(x) = x,\ (fh)^G(x) = f^{\boldsymbol{G}}(x)h^{\boldsymbol{G}}(x),$$

for any $f, h \in E, x \in G$. (For a fixed groupoid $\boldsymbol{G}$ we usually write $f(x)$ instead of $f^{\boldsymbol{G}}(x)$.)

Each $f \in E$ induces a transformation $f^{\boldsymbol{E}}$ of E. We define a new operation " ∘ " on E by:

$$f \circ g = f^{\boldsymbol{E}}(g) = f(g).$$

So, we obtain an algebra $(E,\circ,\cdot)$ with two operations, such that for any $f,g,h\in E$:

$$e\circ f=f\circ e=f$$
$$(fg)\circ h=(f\circ h)(g\circ h).$$

A power $f\in E$ is said to be *irreducible* if $f\neq e$ and $f=g\circ h$ implies $g=e$ or $h=e$.

The following facts for any $f,g,p,q\in E$, $t,u\in F$ can be shown by induction on lengths.

2.1 $|f(t)|=|f||t|$.

2.2 $t\in P(f(t))$.

2.3 $(f(t)=g(u)$ *and* $|t|=|u|)$ *iff* $(f=g$ *and* $t=u)$.

2.4 $(f(t)=g(u)$ *and* $|t|\geq|u|)$ *iff* $(\exists!\,h\in E)\big(t=h(u)$ *and* $g=h(f)\big)$.

2.5 $(E,\circ,e)$ *is a cancellative monoid.*

2.6 *If the length of a power* f *is a prime integer, then the power* f *is irreducible.*

2.7 *If* $f\circ p=g\circ q$ *and* p,q *are irreducible, then* $f=g$ *and* $p=q$.

2.8 *For* $f\neq e$ *there is a unique sequence of irreducible powers* $p_1,p_2,\dots,p_k$ such that

$$f=p_1\circ p_2\circ\ldots\circ p_k\ .$$

2.9 *The monoid* $(E,\circ,e)$ *is a free monoid, with a basis the countable set of irreducible powers.*

For a fixed groupoid power $f\in E$ of length *n* we will write x^n instead of $f(x)$. For $n=2$ there is only one power, $x^2=x\cdot x$, but for $n\geq 3$ there are different *n*-th powers. For example, $x^3=x^2\cdot x$ and $x^3=x\cdot x^2$ are different powers, and they are the only powers of length three. There are five different powers of length four: $x^4=x^3\cdot x$, $x^4=x^2\cdot x^2$, $x^4=x\cdot x^3$, $x^4=(x\cdot x^2)\cdot x$ and $x^4=x\cdot(x\cdot x^2)$. It is well known that there are $\frac{(2n-2)!}{n!(n-1)!}$ groupoid powers of length *n*, i.e. *n*-th powers (see, for example [2] page 125, or [7] (1.8)).

A CLASS OF GROUPOIDS DETERMINED BY GROUPOID POWERS

Let $f\in E$ be a groupoid power of length *n* and let *B* be a nonempty set. We will present a specific construction of a groupoid, denoted by $\boldsymbol{R}(f,B)$, determined by f and *B*.

If $\boldsymbol{G}=(G,\cdot)$ is a given groupoid, for any nonnegative integer k we define a transformation $(k)\colon x\to x^{(k)}$ of *G* as the *k*-th power of f in the monoid $(E,\circ,e)$, i.e.

$$x^{(0)}=x,\ x^{(k+1)}=f(x^{(k)}).$$

Using the notion x^n instead of f, we have:

$$x^{(0)}=x,\ x^{(k+1)}=\big(x^{(k)}\big)^n.$$

Since a free groupoid $\boldsymbol{F}$ is injective, it follows that the transformation (k) is injective on $\boldsymbol{F}$, for any $k\geq 0$. Thus, for each $k\geq 0$, there exists an injective partial transformation $(-k)\colon x\to x^{(-k)}$ on $\boldsymbol{F}$ defined by:

$$y^{(-k)}=x\ \text{ iff }\ x^{(k)}=y.$$

For any $u\in F$, there exists a largest integer m, such that $u^{(-m)}\in F$. We denote this integer by $[u]$ and call it the *exponent* of u in F.

It is easy to show that the following facts are true for all $u,v\in F$ and all integers t and s.

3.1 $u^{(t)}\in F$ *iff* $t+[u]\geq 0$.

3.2 *If* $t+[u]\geq 0$, *then* $\left|u^{(t)}\right|=n^t|u|$.

3.3 *If* $t+[u]\geq 0$ *and* $t+s+[u]\geq 0$, *then* $\big(u^{(t)}\big)^{(s)}=u^{(t+s)}$.

3.4 *If* $t+[u]\geq 0$ *and* $s-t+[v]\geq 0$, *then* $(u^{(t)}=v^{(s)}$ *iff* $u=v^{(s-t)})$.

Definition 3.1 We define $R(f,B)$, as the least subset of F such that $B\subseteq R(f,B)$ and:

$vw\in R(f,B)$ iff
$[(vw=f(u)$ for some $u\in R(f,B))$ or
$(v,w\in R(f,B)$ and $min\{[v],[w]\}=0)]$.

We will often write R instead of $R(f,B)$.

Let $S=R\backslash\{u^{(1)}=f(u)|u\in R\}$.

Proposition 3.5 *For every* $h\in E$ *and* $x\in F$, $h(x)\in S$ *implies* $x\in R$.

Proof. The proof is by induction on the length $|h|$ of h. For $|h|=1$, $h(x)=x\in S$ implies $x\in R$.

Assume that for any $g\in E$ with $|g|<k$, $g(x)\in S$ implies $x\in R$. Let $h=h_1h_2$ and $|h|=k$. Then $h(x)=h_1(x)h_2(x)\in S\subseteq R$ implies that $h_1(x),h_2(x)\in R$ and $min\,\{[h_1(x)],[h_2(x)]\}=0$, i.e. $[h_i(x)]=0$ for some $i\in\{1,2\}$. This implies that $h_i(x)\in S$, and the inductive hypothesis, since $|h_i|<k$, implies that $x\in R$. ∎

Proposition 3.6 *For every* $u\in F$, $u^{(1)}=f(u)\in R$ *iff* $u\in R$.

Proof. The definition of R implies that, if $u\in R$, then $u^{(1)}=f(u)\in R$.

Let $u^{(1)}\in R$. If $u^{(1)}=v^{(1)}$ for some $v\in R$, then, since the transformation (1) is injective, it follows that $u=v\in R$. If $u^{(1)}\neq v^{(1)}$ for every $v\in R$, i.e. $u^{(1)}\in S$, then Proposition 3.5 implies that $u\in R$. ∎

Proposition 3.7 *If for an integer* t *and* $u\in F$, $t+[u]\geq 0$, *then* $(u^{(t)}\in R$ *iff* $u\in R)$.

Proof. The proof is by induction on t, starting from $-[u]$, using the fact 3.3 and Proposition 3.6. ∎

Proposition 3.8 *For every* $u,v\in R(f,B)$,

$$\left(u^{(-m)}v^{(-m)}\right)^{(m)} \in R(f,B),$$
where $m = min\{[u],[v]\}$.

Proof. The fact that $m = min\{[u],[v]\}$ implies that $-m + [u] \geq 0$ and $-m + [v] \geq 0$, and so, Proposition 3.7 implies that $u^{(-m)}, v^{(-m)} \in R$. Since $m = [u]$ or $m = [v]$, we have $\left[u^{(-m)}\right] = 0$ or $\left[v^{(-m)}\right] = 0$, and the definition of R implies that
$$\left(u^{(-m)}v^{(-m)}\right)^{(m)} \in R(f,B). \blacksquare$$

If for $u, v \in R(f,B)$ we define $u * v$ by:
$$u * v = \left(u^{(-m)}v^{(-m)}\right)^{(m)},$$
where $m = min\{[u],[v]\}$, then $\boldsymbol{R} = (R(f,B),*)$ is a groupoid.

Proposition 3.9 *For every* $u, v \in R(f,B)$,
$$u^{(1)} * v^{(1)} = (u * v)^{(1)}.$$
Proof. If $m = min\{[u],[v]\}$, then
$$\min\{[u^{(1)}],[v^{(1)}]\} = m + 1.$$
The definition of $*$ and the fact 3.3 imply:
$$u^{(1)} * v^{(1)} = \left(\left(u^{(1)}\right)^{(-(m+1))}\left(v^{(1)}\right)^{(-(m+1))}\right)^{(m+1)}$$
$$= \left((u)^{(-m)}(v)^{(-m)}\right)^{(m+1)}$$
$$= \left(\left((u)^{(-m)}(v)^{(-m)}\right)^{(m)}\right)^{(1)}$$
$$= (u * v)^{(1)}. \blacksquare$$

Let $\mathcal{M}$ be a variety of groupoids. If $\boldsymbol{G} \in \mathcal{M}$, we say that $\boldsymbol{G}$ is an $\mathcal{M}$*-groupoid*, and if it is free in $\mathcal{M}$, we say that it is $\mathcal{M}$*-free*.

For a groupoid power $f \in E$, i.e. x^n, we denote by $\mathcal{M}_f$ the variety of all the groupoids satisfying the identity
$$f(xy) = f(x)f(y), \text{ i.e.}$$
$$(xy)^n = x^n y^n.$$

For the groupoid power $e^2 = ee$, i.e. for the groupoid power x^2, we denote $\mathcal{M}_{e^2}$ by $\mathcal{M}_2$.

We state the following theorems, proven in [4] in their original forms.

Theorem 1. $\boldsymbol{R} = (R(e^2,B),*)$ *is* $\mathcal{M}_2$*-free and the set B is the unique basis for* $\boldsymbol{R}$.

Theorem 2. *An* $\mathcal{M}_2$*-groupoid* $\boldsymbol{H} = (H,\cdot)$ *is* $\mathcal{M}_2$*-free iff the following conditions hold.*

(i) *Every divisor chain in* $\boldsymbol{H}$ *is finite.*

(ii) *If* $x^2 = y^2$*, then* $x = y$.

(iii) *If* $xy = uv$, $x \neq y$ *and* $u \neq v$*, then* $x = u$ *and* $y = v$.

(iv) *If* $x^2 = yz$ *and* $y \neq z$*, then there are* u, v *such that* $x = uv$, $y = u^2$ *and* $z = v^2$.

Then the set P of primes in $\boldsymbol{H}$ *is nonempty and the unique basis for* $\boldsymbol{H}$.

Theorem 3. *If* $\boldsymbol{H}$ *is an* $\mathcal{M}_2$*-free groupoid, then there exist subgroupoids* $\boldsymbol{G}, \boldsymbol{Q}$ *of* $\boldsymbol{H}$*, such that* $\boldsymbol{G}$ *is not* $\mathcal{M}_2$*-free, and* $\boldsymbol{Q}$ *is* $\mathcal{M}_2$*-free with an infinite rank.*

In [4], for any positive integer *n*, the groupoid power e^n, i.e. x^n, is defined as follows:
$$e^1 = e, \; e^{k+1} = e^k e, \text{ i.e.}$$
$$x^1 = e, \; x^{k+1} = x^k x \,.$$

For the groupoid power e^n, we denote $\mathcal{M}_{e^n}$ by $\mathcal{M}_n$.

The generalizations of Theorems 1 – 3, are also discussed in [4]. **Theorem 1'** and **Theorem 3'** are the same as Theorem 1 and Theorem 3, where 2 is replaced by *n*. **Theorem 2'** is obtained from Theorem 2 by replacing 2 by *n* and by replacing (ii), (iii) and (iv) by:

(ii') *If* $x^n = y^n$*, then* $x = y$.

(iii') *If* $xy = uv$, $x \neq y^{n-1}$ *and* $u \neq v^{n-1}$*, then* $x = u$ *and* $y = v$.

(iv') *If* $x^n = yz$ *and* $y \neq z^{n-1}$*, then there are* u, v *such that* $x = uv$, $y = u^n$ *and* $z = v^n$.

We note that Theorems 2 and 2' characterize $\mathcal{M}_2$-free and $\mathcal{M}_n$-free groupoids in the same way as Theorem 1.1 characterizes free groupoids.

It is easy to check that if $uv \in R(e^2,B)$, then $u, v \in R(e^2,B)$, but this is not the case for $R(e^n,B)$ when $n \geq 3$. For example, if $b \in B$ and $n = 3$, then
$$b^{(2)} \in \left(b^{(1)}\right)^{(1)} = \left(b^{(1)}\right)^2 \cdot b^{(1)} \in R(e^3,B),$$
but $\left(b^{(1)}\right)^2 \notin R(e^3,B)$.

From now on, for a groupoid power $g \in E$, of length *p*, we will often write: $g^{\boldsymbol{F}}(x) = x^p$ for $x \in F$, and $g^{\boldsymbol{R}}(x) = x_*^p$ for $x \in R(f,B)$.

The following examples will show that in general, for a groupoid power $f \in E$, $\boldsymbol{R} = (R(f,B),*)$ does not have to belong to $\mathcal{M}_f$, and there are $u \in R$ such that $[u_*^n] = 0$, where $f(x) = x^n$.

Example 3.1. Let $f = e^2 \circ ((e^2)^2 e) \in E$ and let $B = \{a\}$. The length of *f* is 10, and we write $f(x) = x^{10} = ((x^2)^2 x)^2 = (x^5)^2 = x^{(1)}$.

Let $u = a^5 = (a^2)^2 a$. Since $a \in B \subseteq R$, and $[a] = 0$, we have that $a^2 \in R$ and $[a^2] = 0$. This implies that $(a^2)^2 \in R$ and $[(a^2)^2] = 0$. Next, we obtain that $(a^2)^2 a \in R$ and $[(a^2)^2 a] = 0$. All this implies that $u \in R$ and $[u] = 0$.

Now, we calculate $u_*^{10} = ((u_*^2)_*^2 * u)_*^2$, as follows:

$u_*^2 = u^2 = (a^5)^2 = a^{10} = a^{(1)}$;

$(u_*^2)_*^2 = a^{(1)} * a^{(1)} = (a * a)^{(1)} = (a^2)^{(1)}$;

$(u_*^2)_*^2 * u = (a^2)^{(1)} * a^5 = (a^2)^{(1)} a^5$; and

$u_*^{10} = (a^2)^{(1)} a^5 * (a^2)^{(1)} a^5 = ((a^2)^{(1)} a^5)^2$.

We see that $[u_*^{10}] = 0$.

Next, let $v = u$. Then:

$(u * v)_*^{10} = (u_*^2)_*^{10} = \left(a^{(1)}\right)_*^{10} = a^{(2)}$; and

$u_*^{10} * v_*^{10} = (((a^2)^{(1)}a^5)^2)^2$.

Thus, $u_*^{10} * v_*^{10} \neq (u * v)_*^{10}$.

Example 3.2. Let $f = e^3 \circ e^2 \circ e^3 \in E$ and let $B = \{a, b\}$. The length of f is 18, and we write $f(x) = x^{18} = ((x^3)^2)^3 = x^{(1)}$. Let $u = ((a^3)^3)^2$. Since $a \in B \subseteq R$, and $[a] = 0$, we have that $a^2 \in R$ and $[a^2] = 0$. This implies that $a^3 = a^2a \in R$ and $[a^3] = 0$. Next, $(a^3)^2 \in R$ and $[(a^3)^2] = 0$. This, together with $a^3 \in R$, implies that $((a)^3)^3 \in R$ and $[(a^3)^3] = 0$, and so, $u \in R$ and $[u] = 0$.

Now, we calculate $u_*^{18} = ((u_*^3)_*^2)_*^3$ as follows:

$u_*^2 = u * u = u^2$ and $[u^2] = 0$;

$u_*^3 = u_*^2 * u = u^2 * u = u^3$

$= (((a^3)^3)^2)^3 = (a^3)^{(1)}$;

$(u_*^3)_*^2 = (a^3)^{(1)} * (a^3)^{(1)} = ((a^3)^2)^{(1)}$;

$((u_*^3)_*^2)_*^2 = ((a^3)^2)^{(1)} * ((a^3)^2)^{(1)}$

$= (((a^3)^2)^2)^{(1)}$;

$((u_*^3)_*^2)_*^3 = (((a^3)^2)^2)^{(1)} * ((a^3)^2)^{(1)}$

$= (((a^3)^2)^3)^{(1)} = \left(a^{(1)}\right)^{(1)} = a^{(2)}$.

We see that $[u_*^{18}] = 2$, while $[u] = 0$.

In the same way, for $v = ((b^3)^3)^2$, we obtain that $v_*^{18} = b^{(2)}$.

The previous calculations imply that

$$u_*^{18} * v_*^{18} = a^{(2)} * b^{(2)} = (ab)^{(2)}.$$

In the calculation of $(u * v)_*^{18}$, we have:

$u * v = uv$; $\quad (u * v)_*^3 = (uv)_*^3 = (uv)^3$;

$((u * v)_*^3)_*^2 = ((uv)^3)_*^2 = ((uv)^3)^2$; and

$(u * v)_*^{18} = (((uv)^3)^2)_*^3 = (((uv)^3)^2)^3$

$= (uv)^{(1)}$.

Since $(ab)^{(2)} \neq (uv)^{(1)}$, it follows that

$$u_*^{18} * v_*^{18} \neq (u * v)_*^{18}.$$

We see that the groupoid powers in the previous examples are not irreducible, and moreover, the groupoid power $x^n = (x^p)^q$ has $(x^q)^2$ as its part, i.e. $(x^q)^2 \in P(x^n)$. That is why we consider a special class of groupoid powers, called simple.

We say that a groupoid power x^n is *complex,* if $x^n = ((x^p)^r)^q$ for some $p, q \geq 2$ and $r \geq 1$, and $P(x^n)$ contains $(x)^q(x^r)^q$ or $(x^r)^q(x)^q$. We say that a power x^n is *simple,* if it is not complex.

Irreducible groupoid powers are simple. Since any power x^n, for a prime *n*, is irreducible, it follows that it is simple.

$\mathcal{M}_f$-FREE GROUPOIDS

Let $f = gh \in E\backslash\{e\}$. For a given groupoid $\boldsymbol{G} = (G,\cdot)$ let $T(f, G) \subseteq G \times G$ be defined as:

$$T(f, G) = \{(g(u), h(u)) | u \in G\}.$$

With the notation $f(x) = x^n = x^px^q$,

$$T(f, G) = \{(u^p, u^q) | u \in G\}.$$

Theorem 4.1 *Let $f = gh$, $g, h \in E\backslash\{e\}$ and with the notation $f(x) = x^n = x^px^q$, let a groupoid $\boldsymbol{H} = (H,\cdot)$ satisfies the following conditions.*

(i) *Every divisor chain in $\boldsymbol{H}$ is finite.*

(ii) *If $x^n = y^n$ in $\boldsymbol{H}$, then $x = y$.*

(iii) *If $xy = uv$ in $\boldsymbol{H}$, and $xy \neq z^n$ for each $z \in G$, then $x = u$ and $y = v$.*

(iv) *If $x^n = yz$ in $\boldsymbol{H}$ and $(y, z) \notin T(f, H)$, then there are $u, v \in H$, so that $x = uv$, $y = u^n$ and $z = v^n$.*

Then, the groupoid $\boldsymbol{H}$ is $\mathcal{M}_f$-free and the set B of primes in $\boldsymbol{H}$ is nonempty and is the unique basis of $\boldsymbol{H}$.

Proof. The proof is almost the same as the proof of Proposition 2.3 from [4], which is in fact Theorem 4.1 for $f = e^2$, i.e. for the power x^2. The only difference is the following.

The conditions (ii), (iii) and (iv), imply that, for the power x^2, any element $u \in H$ has at most three divisors (shown in [4]), while for any other power, any element $u \in H$ has at most four divisors. The proof of this for a power different than x^2 is as follows. Let $u \in H$.

If u is prime, then it has 0 divisors. If u is not prime, we consider two cases.

Case 1. For any $x \in H$, $u \neq x^n$. Then, the condition (iii) implies that u has at most two divisors.

Case 2. For some $x \in H$, $u = x^n = x^px^q$. The condition (ii) implies that the element x is unique. If x is prime and $u = yz$, then $(y, z) \notin T(f, H)$ would imply that there are $v, w \in H$, so that $x = vw$, that is not possible. Hence, for x prime, u has at most two divisors. If x is not prime, i.e. if $x = vw$, then $u = x^n = x^px^q = v^nw^n$, and conditions (ii) and (iv) imply that u has at most four divisors. ■

Theorem 4.2 *If $f(x) = x^n$, and $u_*^n = u^n$ for every $u \in (R(f, B),*)$, then $(R(f, B),*)$ satisfies the conditions* (i) *to* (iv)*, from Theorem 4.2, and so it is $\mathcal{M}_f$-free with basis B.*

Proof. Let $x^n = x^px^q$.

If $x * y = z$, then $|z| > |x|$, $|z| > |y|$, and this implies that $\boldsymbol{R}$ satisfies (i).

If $x_*^n = y_*^n$, then $x^n = y^n$ in $\boldsymbol{F}$, and so $x = y$. Hence, $\boldsymbol{R}$ satisfies (ii).

If $x * y = u * v$ and $x * y \neq z_*^n$ for any $z \in R$, then $\min\{[x],[y]\} = 0 = \min\{[u],[v]\}$. This implies that $x * y = xy$, $u * v = uv$, and $xy = uv$ in $\boldsymbol{F}$. So, $x = u$ and $y = v$. Hence, $\boldsymbol{R}$ satisfies the condition (iii).

Let $x_*^n = y * z$ and $(y,z) \notin T(f,R)$.

If $\min\{[y],[z]\} = 0$, then

$$x^p x^q = x^n = x_*^n = y * z = yz$$

and so, $(y,z) \in T(f,R)$. Hence, $\min\{[y],[z]\} > 0$, and this implies that there are $u, v \in R$, such that $y = u^n = u_*^n$, $z = v^n = v_*^n$, and $x^n = (u * v)^n$, i.e. $x = u * v$. Hence, $\boldsymbol{R}$ satisfies (iv). ■

Theorem 4.3 *Let $f \in E$ be a simple groupoid power, with $f(x) = x^n$. Then, for every $u \in (R(f,B),*)$,*

$$u_*^n = u^n.$$

Proof. By Proposition 3.9 it is enough to consider $x \in R$ with $[x] = 0$. We will show that $x_*^t = x^t$, for any part x^t of x^n.

(1) Since $[x] = 0$, it follows that $x_*^t = x * x = x^2$.

(2) Let $x_*^t = x^t$, for any part x^t of x^n with $t < k$.

(2.1) Let $x^k = x^q x^s$ be a part of x^n with $q < s$.

Then, $x_*^k = x_*^q * x_*^s = x^q * x^s$.

We will show that $\min\{[x^q],[x^s]\} = 0$, which implies that $x_*^k = x^k$. Assume contrary, that, $x^q = u^n$ and $x^s = v^n$ for some $u, v \in R$. Since $[x] = 0$ and $k \leq n$, it follows that $2 \leq q, s < n$. This, implies that, $x = u^m$ and $x = v^p$ for some $m, p \geq 2$, and $u^n = (u^m)^q$, $v^n = (v^p)^s$, and we obtain that

$$z^n = (z^m)^q = (z^p)^s .$$

Since $q < s$, it follows that $z^n = (z^m)^q = (z^p)^s$, $z^m = (z^p)^r$ and $z^s = (z^r)^q$. With all this, we have: $x^n = ((x^p)^r)^q$ and $x^q x^s = x^q (x^r)^q$ is a part of x^n, i.e. the power x^n is not simple. This is a contradiction.

(2.2) The proof that $x_*^k = x^k$, for $x^k = x^s x^q$ with $q < s$ is the same as the proof in (2.1).

(2.3) Let $x^k = x^q x^s$ be a part of x^n with $q = s$, but possibly different powers x^q, x^s, and let $x^q = u^n$ and $x^s = v^n$ for some $u, v \in R$. Similarly as in (2.2), we obtain that, $x = u^m = v^p$, for some $m, p \geq 2$, and $u^n = (u^m)^q$, $v^n = (v^p)^s$. Now, $q = s$ and $sp = n = qm$, imply that $p = m$. This, together with $u^m = v^p$ in $\boldsymbol{F}$ implies that $u = v$ and z^m, z^p are the same powers. Next, $(u^m)^q = (v^p)^s$ in $\boldsymbol{F}$ implies that z^q, z^s are the same powers. All this implies that, $x^n = ((x^p)^1)^q$ and $x^q x^q = x^q (x^1)^q$ is a part of x^n, i.e. x^n is not simple. Hence, $[x^q] = 0$ or $[x^s] = 0$, and $x_*^k = x^k$. ■

The following generalization of Theorem 1 from [4], follows from Theorems 4.2 and 4.3.

Theorem 4.4 *If $f \in E$ is a simple groupoid power, then $(R(f,B),*)$ is $\mathcal{M}_f$-free with basis B, and satisfies the conditions* (i), (ii), (iii) *and* (iv) *from Theorem 4.1.* ■

The next theorem characterizes $\mathcal{M}_f$-free groupoids, for a simple power f, and it is a generalization of Theorem 2 from [4] and Theorem 1.1. Its proof follows from Theorems 4.1, 4.2 and 4.3.

Theorem 4.5 *Let $f \in E$ be a simple groupoid power. A groupoid $\boldsymbol{H} = (H,\cdot)$ is $\mathcal{M}_f$-free if and only if it satisfies the conditions* (i), (ii), (iii) *and* (iv) *from Theorem 4.1. Then, the set B of primes in H is nonempty and is the unique basis of $\boldsymbol{H}$.* ■

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ЗА СЛОБОДНИ ГРУПОИДИ СО $(xy)^n = x^n y^n$

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Во трудовите [3,5,6,8,9], Чупона со соработниците ги истражува слободните објекти во многуобразија групоиди кои задоволуваат некои идентитети меѓу групоидни степени. Слободни објекти во многуобразието групоиди дефинирано со идентитетот $(xy)^2 = x^2y^2$ се разгледувани во трудот [4]. Пред повеќе од 20 години, заедно со професор Чупона, добивме каноничен опис на слободни објекти во многуобразието групоиди кои го задоволуваат идентитетот $(xy)^n = x^n y^n$ за некои групоидни степени x^n. Овој резултат не беше публикуван, а прашањето за наоѓање каноничен опис на слободни групоиди за произволен групоиден степен x^n е сеуште отворено. Во овој труд е дадено мало подобрување на резултатот од пред 20 години, односно е даден каоничен опис на слободни групоиди во многуобразието групоиди дефинирано со идентитетот $(xy)^n = x^n y^n$, за едноставни групоидни степени x^n. За такви степени, слободните групоиди се карактеризирани со помош на инјективните групоиди од тоа многуобразие.

Клучни зборови: многуобразие групоиди, слободен групоид, групоидни степени