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ON A CLASS OF PRESENTATIONS IN VARIETIES OF VECTOR VALUED SEMIGROUPS

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To the memory of Professor Gjorgji Čupona, with deep respect and immense gratitude

We define a special class of (n, m)-semigroup presentations in vector varieties of (n, m)-semigroups and apply previously obtained results on existence of effective reductions within, under certain conditions. As a consequence, good combinatorial descriptions are provided.

Key words: (n, m)-semigroup, (n, m)-presentation, variety, reduction

INTRODUCTION

This work is a continuation of our results presented in [7, 8, 9]. In [10] we have discussed the word problem solvability for some classes of vector (n, m)-presentations. Here we try to apply some of those results for varieties of (n, m)-semigroups, in particular for some classes of vector varieties of (n, m)-semigroups. The introductory notions, basic definitions, and properties are incorporated in the review paper [10], that is our main reference paper. Bellow we annex few additional details necessary for the rest of the text.

- For an (n, m)-presentation of an (n, m)-semigroup $\langle B; \Delta\rangle$ (that is the factor (n, m)-semigroup $\boldsymbol{F(B)}/\overline{\Delta}$ where $\overline{\Delta}$ is the smallest congruence on $\boldsymbol{F(B)}$ such that $\Delta \subseteq \overline{\Delta}$ and $\boldsymbol{F(B)}/\overline{\Delta}$ is an (n, m)-semigroup), it can be easily shown that $\overline{\overline{\Delta}} = \overline{\Delta}$ ([2]).

- Two (n, m)-semigroup presentations $\langle B'; \Delta'\rangle$ and $\langle B''; \Delta''\rangle$ are strictly equivalent if $B' = B''$ and $\overline{\Delta'} = \overline{\Delta''}$. We use the notation $\langle B'; \Delta'\rangle \equiv \langle B''; \Delta''\rangle$ ([3]).

- Given a set of vector (n, m)-relations Δ, we will need to emphasize (in notation) the connection with its corresponding induced binary relations Λ. Thus, we allow elements from B to be represented as $(i, \mathbf{x})$ for some $\mathbf{x} \in B^m$ and $i \in \mathbb{N}_m$. Hence, given $u \in F(B)$ we will also use the notation $(i, u_1^{i-1} u u_{i+1}^m)$ where $i \in \mathbb{N}_m$ and $u_v \in F(B)$ $(v \in \mathbb{N}_m \backslash \{i\})$. In other words, we have the following notation definition:

$$u \in F(B) \iff u = (i, \mathbf{x})$$

for some $i \in \mathbb{N}_m$, $\mathbf{x} = u_1^{m+sk}$, and $s \in \mathbb{N}_0$. (Note that, each element from $F(B)\backslash B$ remains to have a unique representation (i, u_1^{m+sk}) where $i \in \mathbb{N}_m$ and $s \geq 1$). Hence, for vector (n, m)-relations Δ and the corresponding induced binary relations Λ, we will also use the following notation

$$\Delta = \Lambda_\#$$

where

$$\Lambda_\# = \{((i, \mathbf{x}), (i, \mathbf{y})) \mid (\mathbf{x}, \mathbf{y}) \in \Lambda,\, i \in \mathbb{N}_m\}.$$

PRESENTATIONS IN VARIETIES OF (n, m)-SEMIGROUPS

The varieties of (n, m)-semigroups were defined in [2] and also explored in [3, 8, 9]. We recall basic definitions and properties necessary for the rest of the text.

If $\boldsymbol{F(}\mathbb{N}\boldsymbol{)}$ is a free poly-(n, m)-groupoid with a basis $\mathbb{N}$ and $\boldsymbol{Q} = (Q, h)$ is a poly-(n, m)-groupoid, for each $\tau \in F(\mathbb{N})$ there exists a smallest $t \in \mathbb{N}$ such that $\tau \in F(\mathbb{N}_t)$

and τ defines a t-ary operation on Q as follows:

i) If $\tau = j \in \mathbb{N}_t$ and $\mathbf{a} = a_1^t \in Q^t$ then $\tau(\mathbf{a}) = a_j$

ii) If $\tau = (i, \tau_1^{m+sk})$ and $\mathbf{a} = a_1^t \in Q^t$ then $\tau(\mathbf{a}) = h_i(\tau_1(\mathbf{a}) \dots \tau_{m+sk}(\mathbf{a}))$, assuming that $\tau_\nu(\mathbf{a})$ are already defined.

Let $\tau, \omega \in F(\mathbb{N})$. Then $\tau, \omega \in F(\mathbb{N}_t)$ for some $t \in \mathbb{N}$. A poly-(n,m)-groupoid $\boldsymbol{Q}$ satisfies the (n,m)-identity (τ,ω) (i.e. $\boldsymbol{Q} \models (\tau,\omega)$), if $\tau(\mathbf{a}) = \omega(\mathbf{a})$ for an arbitrary $\mathbf{a} = a_1^t \in Q^t$.

A class of (n,m)-semigroups $\mathcal{V}$ is a variety if and only if there exists a set of (n,m)-identities Θ such that $\boldsymbol{G} \models \Theta$ for every $\boldsymbol{G} \in \mathcal{V}$. This means that $\boldsymbol{G} \models (\tau,\omega)$, for every $(\tau,\omega) \in \Theta$ and every $\boldsymbol{G} \in \mathcal{V}$. We use the notation $\mathcal{V} = Var\Theta$.

In [8] we gave a description of the complete system of (n,m)-identities $\widehat{\Theta}$ for a variety $Var\Theta$. We also showed that $\psi_0(\boldsymbol{F(}\mathbb{N}\boldsymbol{)})/\widehat{\Theta}$ is a free object in $Var\Theta$ with basis $\mathbb{N}$ where ψ_0 is the reduction for $\langle \mathbb{N}; \emptyset \rangle$ (for more details on ψ_0, see [10]). In [9] we explored a special class of varieties of (n,m)-semigroups, called vector varieties of (n,m)-semigroups. They are originally defined in [3], as follows:

Let $p = m + sk$, $q = m + rk$, where $s, r \geq 0$ and let $(i_1^p, j_1^q) \in \mathbb{N}^+ \times \mathbb{N}^+$. An (n,m)-semigroup $\boldsymbol{G} = (G; g)$ satisfies the vector (n,m)-identity (i_1^p, j_1^q) (i.e. $\boldsymbol{G} \models (i_1^p, j_1^q)$), if $g(a_{i_1} \dots a_{i_p}) = g(a_{j_1} \dots a_{j_q})$ for an arbitrary $a_1^t \in G^t$, where $t = \max_{\mu,\nu}\{i_\mu, j_\nu\}$.

Every vector (n,m)-identity (i_1^p, j_1^q) induces a set of (n,m)-identities $(i_1^p, j_1^q)_\# \subseteq \psi_0(F(\mathbb{N})) \times \psi_0(F(\mathbb{N}))$ defined by: $(i_1^p, j_1^q)_\# = \{((i, i_1^p), (i, j_1^q)) \,|\, i \in \mathbb{N}_m\}$, and moreover, $\boldsymbol{G} \models (i_1^p, j_1^q) \iff \boldsymbol{G} \models (i_1^p, j_1^q)_\#$. Consequently, if Θ' is a set of vector (n,m)-identities then it induces a set of (n,m)-identities $\Theta'_\#$, and, $\boldsymbol{G} \models \Theta' \iff \boldsymbol{G} \models \Theta'_\#$.

Definition 2.1 A variety of (n,m)-semigroups $\mathcal{V}$ is called a vector variety of (n,m)-semigroups, if there exists a set of vector (n,m)-identities $\Theta'_\#$ such that $\mathcal{V} = Var\Theta'_\#$.

In continuation we will define (n,m)-semigroup presentations in varieties of (n,m)-semigroups. The main idea arises from [3].

Let Θ be a set of (n,m)-identities and let $\boldsymbol{F(B)} = (F(B); f)$ be a free poly-(n,m)-groupoid with basis $B \neq \emptyset$. Every (n,m)-identity $(\tau,\omega) \in F(\mathbb{N}_t) \times F(\mathbb{N}_t)$ defines a relation on $F(B)$ given by

$(\tau,\omega)(F(B)) = \{(\tau(u_1^t), \omega(u_1^t)) \,|\, u_1^t \in F(B)^t\}$

Thus, Θ defines a corresponding set $\Theta(F(B)) \subseteq F(B) \times F(B)$ given by

$$\Theta(F(B)) = \bigcup_{(\tau,\omega)\in\Theta} (\tau,\omega)(F(B)) =$$
$$\{\, (\tau(u_1^t), \omega(u_1^t)) \,|\, (\tau,\omega) \in \Theta,$$
$$\tau, \omega \in F(\mathbb{N}_t),\ u_1^t \in F(B)^t,\ t \in \mathbb{N} \,\}.$$

Clearly, $\Theta(F(B))$ is a set of (n,m)-defining relations on B.

The following result is stated in [3], here we give its proof.

Proposition 2.1 *$\langle B; \Theta(F(B)) \rangle$ is a free object in $Var\Theta$ with basis B.*

Proof. Recall that $\langle B; \Theta(F(B)) \rangle = \boldsymbol{F(B)}/\overline{\Theta(F(B))}$ where $\overline{\Theta(F(B))}$ is the smallest congruence on $\boldsymbol{F(B)}$ such that $\Theta(F(B)) \subseteq \overline{\Theta(F(B))}$ and $\boldsymbol{F(B)}/\overline{\Theta(F(B))}$ is an (n,m)-semigroup. Let $(\tau,\omega) \in \Theta$. Then $\tau, \omega \in F(\mathbb{N}_t)$ for some $t \in \mathbb{N}$. For an arbitrary sequence $u_1^{\overline{\Theta(F(B))}}, \dots, u_t^{\overline{\Theta(F(B))}}$ from $F(B)/\overline{\Theta(F(B))}$, we have

$$\tau(u_1^{\overline{\Theta(F(B))}} \dots u_t^{\overline{\Theta(F(B))}}) = (\tau(u_1^t))^{\overline{\Theta(F(B))}} = (\omega(u_1^t))^{\overline{\Theta(F(B))}} = \omega(u_1^{\overline{\Theta(F(B))}} \dots u_t^{\overline{\Theta(F(B))}}).$$

Thus, $\boldsymbol{F(B)}/\overline{\Theta(F(B))} \models (\tau,\omega)$. Hence, $\boldsymbol{F(B)}/\overline{\Theta(F(B))} \models \Theta$ and therefore $\boldsymbol{F(B)}/\overline{\Theta(F(B))} \in Var\Theta$. It is clear that $\overline{\Theta(F(B))}$ is the smallest congruence on $\boldsymbol{F(B)}$ containing $\Theta(F(B))$ and such that $\boldsymbol{F(B)}/\overline{\Theta(F(B))} \in Var\Theta$, and thus we conclude that $\boldsymbol{F(B)}/\overline{\Theta(F(B))}$ is a free object in $Var\Theta$. Namely, for arbitraries $\boldsymbol{Q} \in Var\Theta$ and $\xi : B \to Q$, there is a unique homomorphic extension $\bar{\xi} : \boldsymbol{F(B)} \to \boldsymbol{Q}$ and moreover, $\boldsymbol{F(B)}/\ker\bar{\xi} \in Var\Theta$. The fact that $\overline{\Theta(F(B))}$ is the smallest congruence on $\boldsymbol{F(B)}$ such that $\boldsymbol{F(B)}/\overline{\Theta(F(B))}$ is in $Var\Theta$, implies that $\overline{\Theta(F(B))} \subseteq \ker\bar{\xi}$. Therefore, we define a map $\eta : F(B)/\overline{\Theta(F(B))} \to Q$, by: $\eta(u^{\overline{\Theta(F(B))}}) = \bar{\xi}(u)$. It is straightforward to check that η is a homomorphism, since $\bar{\xi}$ is a homomorphism, and

$\eta(nat(\overline{\Theta(F(B))})_{|B}) = \bar{\xi}_{|B} = \xi$. Also, η is unique, since $\bar{\xi}$ is unique. □

From now on, the congruence $\overline{\Theta(F(B))}$ will be denoted by $\overline{\Theta}$ and consequently, $\boldsymbol{F(B)}/\overline{\Theta(F(B))} = \boldsymbol{F(B)}/\overline{\Theta}$.

For a given $\Delta \subseteq F(B) \times F(B)$, we have $\Delta \cup \Theta(F(B)) \subseteq F(B) \times F(B)$, that is a set of (n,m)-defining relations on B, and thus $\langle B;\ \Delta \cup \Theta(F(B))\rangle$ is an (n,m)-presentation of an (n,m)-semigroup.

Definition 2.2 For given B, Θ, and Δ, we denote the (n,m)-semigroup presentation $\langle B;\ \Delta \cup \Theta(F(B))\rangle$ by $\langle B;\ \Delta;\ \Theta\rangle$, and we say that $\langle B;\ \Delta;\ \Theta\rangle$ is a presentation of an (n,m)-semigroup in the variety $Var\Theta$.

In particular, we define vector (n,m)-semigroup presentations in (vector) varieties of (n,m)-semigroups.

Definition 2.3 $\langle B;\ \Delta;\ \Theta\rangle$ is a vector presentation of an (n,m)-semigroup in $Var\Theta$, if $\langle B;\ \Delta\rangle$ and $\langle \mathbb{N};\ \Theta\rangle$ are vector (n,m)-presentations.

Thus, and by the notation given in the introduction part, given a vector (n,m)-semigroup presentation $\langle B;\ \Delta;\ \Theta\rangle$ in $Var\Theta$ we can also denote it as $\langle B;\ \Lambda;\ \Theta'\rangle$, where:

$\Lambda \subseteq B^+ \times B^+$ and $\Delta = \Lambda_\#$;

$\Theta' \subseteq \mathbb{N}^+ \times \mathbb{N}^+$ and $\Theta = \Theta'_\#$.

Given $\langle B;\ \Lambda;\ \Theta'\rangle$, the set of vector (n,m)-identities $\Theta' \subseteq \mathbb{N}^+ \times \mathbb{N}^+$ induces a set $\Theta'(B) \subseteq B^+ \times B^+$ defined by:

$(a_1^p, c_1^q) \in \Theta'(B)$ if there exist $(i_1^p, j_1^q) \in \Theta'$ and a sequence $b_1, b_2, \ldots \in B$ such that

$a_\mu = b_{i_\mu}$, $\mu \in \mathbb{N}_p$ and $c_\upsilon = b_{j_\upsilon}$, $\upsilon \in \mathbb{N}_q$.

In other words,

$$\Theta'(B) = \{(b_{i_1}\ldots b_{i_p},\ b_{j_1}\ldots b_{j_q}) \mid (i_1^p, j_1^q) \in \Theta',\ b^t \in B^t,\ t = \max_{\mu,\upsilon}\{i_\mu, j_\upsilon\}\}.$$

Now, $\Lambda \cup \Theta'(B) \subseteq B^+ \times B^+$ is a set of vector (n,m)-relations on B, and thus $\langle B; \Lambda \cup \Theta'(B)\rangle$ is a vector (n,m)-presentation of an (n,m)-semigroup. But, $\langle B; \Lambda \cup \Theta'(B)\rangle$ is not in $Var\Theta'_\#$ in general case.

Example 2.1. Let $n = 3$, $m = 2$, $B = \{a,b\}$, $\Lambda = \emptyset$, and let Θ' be a set of $(3,2)$-identities defined by:

$\Theta' = \{(l^3, l^2)\}$ for some $l \in \mathbb{N}$, i.e.

$$\Theta'_\# = \{((1,lll),(1,ll)),((2,lll),(2,ll))\} = \{((1,lll),l),((2,lll),l)\}.$$

We have that $\langle a,b;\ \Theta'\rangle = \langle a,b;\ \Theta'_\#\rangle$ is a $(3,2)$-semigroup presentation in $Var\Theta'_\#$. Moreover, the $(3,2)$-semigroup $\langle a,b;\ \Theta'_\#\rangle = \boldsymbol{F(a,b)}/\overline{\Theta'_\#(F(a,b))}$ is a free object in $Var\Theta'_\#$ with basis $\{a,b\}$. On the other hand, the $(3,2)$-semigroup presentation $\langle B;\ \Theta'(B)\rangle = \langle a,b;\ \Theta'(a,b)\rangle$ represents the $(3,2)$-semigroup $\boldsymbol{F(a,b)}/\overline{(\Theta'(a,b))_\#}$. It is easy to see that $(\Theta'(a,b))_\# \subseteq \Theta'_\#(F(a,b))$ and thus $\overline{(\Theta'(a,b))_\#} \subseteq \overline{\Theta'_\#(F(a,b))}$. Consequently, if two elements are equal in $\langle a,b;\ \Theta'(a,b)\rangle$, they are equal in $\langle a,b;\ \Theta'\rangle$ as well, The opposite is not true. For example, $(2,\ (1,\ aba)(1,\ aba)(1,\ aba)) = (1,\ aba)$ in $\langle a,b;\ \Theta'\rangle$ but $(2,\ (1,\ aba)(1,\ aba)(1,\ aba)) \neq (1,\ aba)$ in $\langle a,b;\ \Theta'(a,b)\rangle$. We conclude that $\langle a,b;\ \Theta'(a,b)\rangle \notin Var\Theta'_\#$. □

Proposition 2.2
$\langle B;\ \Lambda;\ \Theta'\rangle \equiv \langle B; \Lambda \cup \Theta'(B)\rangle$ if and only if $\langle B; \Lambda \cup \Theta'(B)\rangle \in Var\Theta'_\#$.

Proof. $(\Rightarrow)$. Straightforward.
$(\Leftarrow)$. It is easy to notice that

$(\Lambda_\# \cup \Theta'(B)_\#) \subseteq (\Lambda_\# \cup \Theta'_\#(F(B)))$,

and thus $\overline{\Lambda_\# \cup \Theta'(B)_\#} \subseteq \overline{\Lambda_\# \cup \Theta'_\#(F(B))}$.

Since $\boldsymbol{F(B)}/\overline{\Lambda_\# \cup \Theta'(B)_\#} \in Var\Theta'_\#$, it follows that for $i \in \mathbb{N}_m$, for an (n,m)-identity $(i_1^{p'}, j_1^{q'}) \in \Theta'$, and for a sequence u_1^t from $F(B)$, where $t = \max_{\mu,\upsilon}\{i_\mu, j_\upsilon\}$:

$$f_i(u_{i_1}^{\overline{\Lambda_\# \cup \Theta'(B)_\#}} \ldots u_{i_{p'}}^{\overline{\Lambda_\# \cup \Theta'(B)_\#}}) = f_i(u_{j_1}^{\overline{\Lambda_\# \cup \Theta'(B)_\#}} \ldots u_{j_{q'}}^{\overline{\Lambda_\# \cup \Theta'(B)_\#}}),$$

i.e. $(f_i(u_{i_1}\ldots u_{i_{p'}}))^{\overline{\Lambda_\# \cup \Theta'(B)_\#}} = (f_i(u_{j_1}\ldots u_{j_{q'}}))^{\overline{\Lambda_\# \cup \Theta'(B)_\#}}$.

This implies that

$$\Big((i, u_{i_1}^{i_{p'}}), (i, u_{j_1}^{j_{q'}})\Big) \in \overline{\Lambda_\# \cup \Theta'(B)_\#}$$

and thus

$\Theta'_\#(F(B)) \subseteq \overline{\Lambda_\# \cup \Theta'(B)_\#}$.

Consequently,

$\Lambda_\# \cup \Theta'_\#(F(B)) \subseteq \overline{\Lambda_\# \cup \Theta'(B)_\#}$,

and moreover,

$\overline{\Lambda_\# \cup \Theta'_\#(F(B))} \subseteq \overline{\Lambda_\# \cup \Theta'(B)_\#}$.

Hence, $\langle B;\ \Lambda;\ \Theta'\rangle \equiv \langle B; \Lambda \cup \Theta'(B)\rangle$. □

Consider now, vector (n,m)-presentations of type $\langle B; \Lambda \cup \Theta'(B)\rangle$.

Since $\langle B; \Lambda \cup \Theta'(B)\rangle$ is a vector (n,m)-presentation of an (n,m)-semigroup, it induces a corresponding binary semigroup presentation, for which we can apply Theorem 4.1, Theorem 4.2, Theorem 4.3, Theorem 4.4 from [10]. As a consequence, and providing that Proposition 2.2 is satisfied, we would get good combinatorial descriptions for $\langle B;\, \Lambda;\, \Theta'\rangle$, that are objects in $Var\Theta'_{\#}$. Moreover, we would have word problem solvability for those vector (n,m)-semigroup presentations in such varieties.

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ЗА ЕДНА КЛАСА ПРЕТСТАВУВАЊА ВО МНОГУОБРАЗИЈА ВЕКТОРСКО ВРЕДНОСНИ ПОЛУГРУПИ

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Во спомен на професор Ѓорѓи Чупона, со длабока почит и огромна благодарност

Дефинираме специјална класа векторски (n,m)-претставувања во векторски многуобразија (n,m)-полугрупи, каде аплицираме претходно добиени резултати за постоење на ефективни редукции, под одредени услови. Како последица, се добиваат добри комбинаторни описи на разгледуваните објекти.

Клучни зборови: (n,m)-полугрупа, (n,m)-претставување, (n,m)-многуобразие, редукција