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Review

ON REDUCTIONS FOR PRESENTATIONS OF VECTOR VALUED SEMIGROUPS: OVERVIEW AND OPEN PROBLEMS

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Dedicated to our beloved Professor Gjorgji Čupona

Finding a satisfactory combinatorial description of an (n,m)-semigroup given by its (n,m)-presentation $\langle B;\Delta\rangle$ is a quite difficult and complex problem. The majority of results obtained so far consider some particular cases or they relate to a special class of presentations of (n,m)-semigroups called vector (n,m)-presentations of (n,m)-semigroups. This is because vector (n,m)-presentations of (n,m)-semigroups induce corresponding binary semigroup presentations, and the question of the existence of a good combinatorial description for $\langle B;\Delta\rangle$ is closely related to the question of the existence of a good combinatorial description for the corresponding induced binary semigroup $\langle B;\Lambda\rangle$. An expository overview of the obtained results is given. We classify conditions under which a good combinatorial description for $\langle B;\Lambda\rangle$ implies word problem solvability for $\langle B;\Delta\rangle$. Furthermore, we state a couple of open problems and consider the application of this ideas in varieties of (n,m)-semigroups, giving suggestions for future investigations.

Key words: (n,m)-semigroup, (n,m)-presentation, reduction, word problem

INTRODUCTION

The development of the theory of multivariable groups (called most often n-ary groups or just n-groups) was initiated, we might say, by the paper [22] of E. L. Post. Later, several authors have made generalizations for n-ary semigroups, semigroups of transformations and algebras of multiplace functions (see for example [16, 17, 23, 24, 25]). Motivated partly by some of these papers, Gj. Čupona and B. Trpenovski have introduced in [1, 30] the notion of an (n,m)-semigroup, that is, a set having a multivariable vector valued associative operation. In continuation we present their definition. The set of positive integers will be denoted by $\mathbb{N}=\{1,2,3,\ldots\}$ and the set of the first t positive integers will be denoted by $\mathbb{N}_t=\{1,2,3,\ldots,t\}$. Moreover, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and $\mathbb{N}_{t,0}=\mathbb{N}_t\cup\{0\}$. For a given set $Q\neq\emptyset$, and $t\in\mathbb{N}$, let Q^t be the cartesian product of t copies of Q. If $\mathbf{x}=(a_1,a_2,\ldots,a_t)\in Q^t$, then we write $\mathbf{x}=a_1^t$, and moreover we identify $\mathbf{x}$ with the word $a_1a_2\ldots a_t$. For such an $\mathbf{x}$ we say that its length $|\mathbf{x}|$ is t. Let Q^+ be the union of all the cartesian products Q^t, $t\in\mathbb{N}$, which is, by the above identification, the free semigroup generated by Q. Let $n,m,k\in\mathbb{N}$, $n=m+k$ be given. A map $f: Q^n\to Q^m$ is called an (n,m)-operation on Q, and (Q,f) is called an (n,m)-groupoid, i.e. a vector valued groupoid. An (n,m)-groupoid (Q,f) is called an (n,m)-semigroup (vector valued semigroup), if the (n,m)-operation is associative, i.e. if $f(\mathbf{x}f(\mathbf{y})\mathbf{z})=f(\mathbf{u}f(\mathbf{v})\mathbf{w})$, for any $\mathbf{xyz}=\mathbf{uvw}\in Q^{n+k}$, $\mathbf{y},\mathbf{v}\in Q^n$. An (n,m)-semigroup (Q,f) is called an (n,m)-group if for each $\mathbf{a}\in Q^k$, $\mathbf{b}\in Q^m$ the equations $f(\mathbf{x}\,\mathbf{a})=\mathbf{b}=f(\mathbf{a}\,\mathbf{y})$ have solutions $\mathbf{x},\mathbf{y}\in Q^m$. For $m=1=k$, the above notions are the usual notions of binary groupoids, semigroups and groups, and for $m=1,k>1$ they are the notions of n-groupoids, n-semigroups and n-groups.

From now on and throughout the paper, we assume that $m \geq 2$.

An (n,m)-groupoid (Q,f) can be considered as an algebra with m n-ary operations $f_1, f_2, \ldots, f_m : Q^n \to Q$, such that $f(\mathbf{x}) = f_1(\mathbf{x})f_2(\mathbf{x})\ldots f_m(\mathbf{x})$. These operations $f_1, f_2, \ldots, f_m$ are called component operations for the (n,m)-operation f. In general, for an associative (n,m)-operation f, the m n-ary operations obtained from f, do not have to be associative. Hence, there is a big difference between studying (n,m)-semigroups and n-semigroups. Vector valued algebraic structures are a generalization of $(2,1)$ structures, and thus they are similar to the binary structures on one hand, but on the other hand they incorporate new ideas and specific properties.

The theory of vector valued structures defined as above, has started to developed in the 80's of the last century. Leaded by one of its founders, Gj. Čupona, a group of macedonian algebraists in just a couple of years gave significant results within this topic. The majority of them are fully cited in the expository papers [2, 4] (though some of them will be referenced separately throughout the paper). Since then, various investigations have been published and upgrades have been made. (See for example [3, 6, 8, 9, 10, 11, 12, 13, 14, 26, 27, 28, 29]). However, some of the ideas given at the very beginning are still looking to be seriously considered and deeply investigated. We do not intend (nor can) state all the results on vector valued semigroups that have been given so far (worldwide). Our aim is to expose the development of the combinatorial theory of (n,m)-semigroups emphasizing the major results on word problem solvability for various classes of (n,m)-semigroups given by their (n,m)-presentations.

The word problem for free vector valued semigroups and groups is solved in [7, 9, 10, 11] by using good combinatorial descriptions for free vector valued semigroups and groups. There have also been made a link between a special class of presentations of (n,m)-semigroups, called vector presentations of (n,m)-semigroup and binary semigroups. They are induced one by another, which will be exposed later in this paper. The majority of the results in continuation concern good combinatorial descriptions that have been obtained for some classes of vector presentations of (n,m)-semigroups. Exploring the above, we have also obtained interesting results for varieties of (n,m)-semigroups. A couple of open problems and suggestions for future investigations will be given at the end.

PRELIMINARIES. PRESENTATIONS OF (n,m)-SEMIGROUPS

The following basic notions were originally given in [2] and [4].

For a given set Q, let

$Q^{m,k} = \{\mathbf{x} | \mathbf{x} \in Q^+, |\mathbf{x}| = m + sk, s \in \mathbb{N}\}$.

If (Q,f) is an (n,m)-semigroup, because of the associative law, the operation f can be extended to an operation, denoted by the same letter, $f : Q^{m,k} \to Q^m$, such that for each $\mathbf{xyz} \in Q^{m,k}$ and $\mathbf{y} \in Q^{m,k}$, $f(\mathbf{x}f(\mathbf{y})\mathbf{z}) = f(\mathbf{xyz})$. As mentioned above, an (n,m)-groupoid (Q,f) can be considered as an algebra with m n-ary operations, $f_j : Q^n \to Q$, $j \in \mathbb{N}_m$. These operations can be extended to an infinite family of operations $f_{j,s} : Q^{m+sk} \to Q$ for $s \in \mathbb{N}$, where for a given s, there are more than one operation $f_{j,s}$. When (Q,f) is an (n,m)-semigroup, for each $s \in \mathbb{N}$, there is only one operation $f_{j,s} : Q^{m+sk} \to Q$ whose union is a map $f_j : Q^{m,k} \to Q$. This leads to the following slightly more general notions. A map $g : Q^{m,k} \to Q^m$ is called a poly-(n,m)-operation and the structure $\boldsymbol{Q} = (Q,g)$ is called a poly-(n,m)-groupoid. A poly-(n,m)-groupoid $\boldsymbol{Q} = (Q,g)$ is called a poly-(n,m)-semigroup if for each $\mathbf{xyz} \in Q^{m,k}$ and $\mathbf{y} \in Q^{m,k}$, $g(\mathbf{x}g(\mathbf{y})\mathbf{z}) = g(\mathbf{xyz})$. There is no essential difference between studying (n,m)-semigroups or poly-(n,m)-semigroups because of the General Associative Law (see [2]). Similarly as above, a poly-(n,m)-groupoid (Q,g) can be considered as an algebra with m poly-n-ary operations, $g_1, g_2, \ldots g_m : Q^{m,k} \to Q$ where $g(\mathbf{x}) = g_1(\mathbf{x})g_2(\mathbf{x})\ldots g_m(\mathbf{x})$. It is easy to see that the usual notions of universal algebra (i.e. free algebras, varieties) can be extended to (n,m)-semigroups.

Let B be a nonempty set and let $\boldsymbol{B}^+$ be the free semigroup with base B. Let $\Lambda \subseteq B^+ \times B^+$. The pair $\langle B; \Lambda \rangle$ is a presentation of the semigroup $\boldsymbol{B}^+/\overline{\Lambda}$ where $\overline{\Lambda}$ is the smallest congruence on $\boldsymbol{B}^+$ containing Λ. We use the notation $\langle B; \Lambda \rangle = \boldsymbol{B}^+/\overline{\Lambda}$. A reduction for $\langle B;\Lambda\rangle$ is a map assigning a chosen element of a congruence class in $\boldsymbol{B^+}/\overline{\Lambda}$ to every element of the congruence class. In order to extend this to (n,m)-semigroups there have been defined a poly-(n,m)-groupoid, $\boldsymbol{F(B)} = (F(B), f)$, with a base B which is an analogy to the free semigroup B^+ above. Its existence follows from the fact that it is an algebra of type $\Omega = \{\omega_r^j | j \in \mathbb{N}_m, r \in \mathbb{N}\}$. We recall its canonical form. (For more details see [7, 14, 34]).

$B_0 = B$,
$B_{p+1} = B_p \cup (\mathbb{N}_m \times B_p^{m,k})$,
$F(B) = \bigcup_{p\geq 0} B_p$.

By choosing different letters, if necessary, for the elements of B, we will have that no element of B is of the form $(j, \mathbf{x})$. The poly-(n,m)-operation f on $F(B)$ is defined by $f(\mathbf{x}) = (1,\mathbf{x})(2,\mathbf{x})\dots(m,\mathbf{x})$. Hierarchy of the elements of $F(B)$ is a map $\chi : F(B) \to \mathbb{N}_0$ defined by $\chi(u) = \min\{p \,|\, u \in B_p\}$, $u \in F(B)$. Clearly, $\chi(u) = p \Leftrightarrow u \in B_p \backslash B_{p-1}$.

The norm on $F(B)$ is a map $\|\ \| : F(B) \to \mathbb{N}$ defined by induction on χ:

$\|u\| = 1$ for $u \in B_0$,
$\|(i, u_1^{m+sk})\| = \|u_1\| + \ldots + \|u_{m+sk}\|$
for $(i, u_1^{m+sk}) \in B_{p+1}\backslash B_p$.

Thus, the norm $\|(i, u_1^{m+sk})\|$ is the number of appearances of elements from B in (i, u_1^{m+sk}).

For $\mathbf{x} \in F(B)^+$ and $\mathbf{x} = x_1^r$, we define the norm as $\|\mathbf{x}\| = \|x_1\| + \ldots + \|x_r\|$.

We note that the elements of $F(B)$ can be also treated as special words over the alphabet $A = B \cup \mathbb{N}_m \cup \{(\} \cup \{,\} \cup \{)\}$. Hence, every $u \in F(B)$ can be considered as an element of A^+ as well.

For a set $\Delta \subseteq F(B) \times F(B)$, we say that Δ is a set of (n,m)-defining relations on B and the pair $\langle B;\Delta\rangle$ is an (n,m)-presentation of an (n,m)-semigroup. We also say that $\langle B;\Delta\rangle$ is an (n,m)-semigroup presentation. The (n,m)-semigroup whose presentation is $\langle B;\Delta\rangle$ is the factor (n,m)-semigroup $\boldsymbol{F(B)}/\overline{\Delta}$ where $\overline{\Delta}$ is the smallest congruence on $\boldsymbol{F(B)}$ such that $\Delta \subseteq \overline{\Delta}$ and $\boldsymbol{F(B)}/\overline{\Delta}$ is an (n,m)-semigroup. We use the notation $\langle B;\Delta\rangle = \boldsymbol{F(B)}/\overline{\Delta}$. The explicit description of $\overline{\Delta}$ and its properties are given in [4]. Given an (n,m)-presentation $\langle B;\Delta\rangle$ of an (n,m)-semigroup, we are interested in the structure of this (n,m)-semigroup. Analogous to the binary case, a reduction for an (n,m)-presentation $\langle B;\Delta\rangle$ is a map assigning a chosen element of a congruence class in $\boldsymbol{F(B)}/\overline{\Delta}$ to every element of the congruence class.

Proposition 2.1 [4] *A map $\psi : F(B) \to F(B)$ is a reduction for the (n,m)-presentation $\langle B;\Delta\rangle$ if and only if the following properties are satisfied*

(i) $(u,v) \in \Delta \Rightarrow \psi(u) = \psi(v)$

(ii) $\psi(i, \mathbf{x}'(1,\mathbf{y})(2,\mathbf{y})\dots(m,\mathbf{y})\mathbf{x}'') = \psi(i, \mathbf{x}'\mathbf{y}\mathbf{x}'')$

(iii) $\psi(i, \mathbf{x}'w\mathbf{x}'') = \psi(i, \mathbf{x}'\psi(w)\mathbf{x}'')$

(iv) $u \;\overline{\Delta}\; \psi(u)$

for all $u, v, w, (i, \mathbf{x}'w\mathbf{x}'')$, $(i, \mathbf{x}'(1,\mathbf{y})(2,\mathbf{y})\dots(m,\mathbf{y})\mathbf{x}'') \in F(B)$. □

We say that $\psi(u)$ is the reduced represent (reduct) for u.

The axiom of choice implies that for any (n,m)-presentation $\langle B;\Delta\rangle$ there exist a reduction. If ψ is a reduction for $\langle B;\Delta\rangle$ such that for any $u \in F(B)$ the reduced represent $\psi(u)$ can be calculated in finitely many steps, ψ is said to be a good (effective) reduction for $\langle B;\Delta\rangle$ and it provides a good combinatorial description for the corresponding (n,m)-semigroup (presented by) $\langle B;\Delta\rangle$.

Proposition 2.2 [4] *A reduction $\psi : F(B) \to F(B)$ for an (n,m)-semigroup presentation $\langle B;\Delta\rangle$ is a homomorphism from $\boldsymbol{F(B)}$ to $(\psi(F(B)); g)$ where*

$\psi(F(B)) = \{u \in F(B) \,|\, \psi(u) = u\}$ *and*
$g(u_1^{m+sk}) = v_1^m \Leftrightarrow v_i = \psi(i, u_1^{m+sk})$, $i \in \mathbb{N}_m$.

Moreover, $\ker\psi = \overline{\Delta}$ *and* $\langle B;\Delta\rangle = (\psi(F(B)), g)$. □

When $\Delta = \emptyset$, then $\langle B;\emptyset\rangle$ is the presentation of the free (n,m)-semigroup generated by B. In 1986, D. Dimovski constructed a canonical form of a free (n,m)-semigroup $S(B)$ generated by B. This was a starting point towards development of a combinatorial theory of vector valued semigroups. Due to its importance, we recall here its construction. (For more details see [7, 14]). We define a map $\psi_0 : F(B) \to F(B)$, by induction on the norm as follows:

(a) $\psi_0(b) = b$, $b \in B$;

(b) Let $u = (i, u_1^{m+sk}) \in F(B)$ and assume that $\psi_0(v) \in F(B)$ is already defined and $\psi_0(v) \neq v$ implies $\|\psi_0(v)\| < \|v\|$ for all $v \in F(B)$, with $\|v\| < \|u\|$. Then

$v_\lambda = \psi_0(u_\lambda)$ is well defined for all $\lambda \in \mathbb{N}_{m+sk}$ and thus $v = (i, v_1^{m+sk}) \in F(B)$.

(b1) If there exists a $\lambda' \in \mathbb{N}_{m+sk}$ such that $v_{\lambda'} \neq u_{\lambda'}$ then $\|v\| < \|u\|$ and

$$\psi_0(u) = \psi_0(v).$$

(b2) If $v_\lambda = u_\lambda$ for all $\lambda \in \mathbb{N}_{m+sk}$ and if $u = (i, u_1^j(1,\mathbf{x})...(m,\mathbf{x})u_{j+m+1}^{m+sk})$ where $\mathbf{x} \in F(B)^{m,k}$ and j is the smallest such index, then

$$\psi_0(u) = \psi_0(i, u_1^j \mathbf{x} u_{j+m+1}^{m+sk}).$$

(b3) If u satisfies neither (b1) nor (b2), then $\psi_0(u) = u$.

Proposition 2. 3 [7, 14] *The map ψ_0 is a good reduction for $\langle B; \emptyset\rangle$.* □

Proposition 2.2 together with Proposition 2.3 imply that $(\psi_0(F(B)), g) = (S(B), g)$ is a free (n,m)-semigroup generated by B. By induction on the norm, we say that an element $u = (i, u_1^{m+sk}) \in F(B)$ is reducible if u_j is reducible for some j, or if $u = (i, u_1^j(1,\mathbf{x})\dots(m,\mathbf{x})u_{j+m+1}^{m+sk})$. Otherwise we say that u is irreducible. With this notion, $S(B)$ is the set of all the irreducible elements in $F(B)$.

The construction above opened new investigation possibilities: To obtain good combinatorial descriptions for various $\langle B; \Delta\rangle$ and to explore the circumstances under which it might be possible. The common approach is to manage to construct a good reduction for $\langle B; \Delta\rangle$ (if possible), a task that is usually quite complicated to achieve. A couple of results on good combinatorial descriptions for some particular $\langle B; \Delta\rangle$ can be found in [4]. In [32] there have been defined a sequence of (n,m)-semigroup presentations $\langle B; \Delta_p\rangle$, for which good reductions have been constructed and consequently, good combinatorial description for such (n,m)-semigroups have been obtained. In [33] we have constructed good reductions for a class of (n,m)-presentations of (n,m)-semigroups that incorporate binary relations within the corresponding (n,m)-relations Δ, under certain conditions. Namely, given a semigroup presentation $\langle B; \Lambda\rangle$ with a good reduction φ that satisfies a pair of conditions, we have defined an associated (n,m)-semigroup presentation $\langle B; \Delta\rangle$ and derived a good reduction ψ for $\langle B; \Delta\rangle$. As a consequence, good combinatorial description of the corresponding (n,m)-semigroup has been given. All this led to a conclusion that valuable results might be obtained by linking (n,m)-semigroup presentations and binary semigroups presentations. In [4], the authors have defined a special class of (n,m)-semigroup presentations, closely related to binary semigroups presentations, called vector (n,m)-presentations of (n,m)-semigroups, and thus made this idea possible. A set of vector (n,m)-defining relations induces also a set of binary relations, i.e. a presentation of a binary semigroup. Under certain conditions for this binary semigroup presentation, there have been obtained good combinatorial descriptions for various classes of (n,m)-semigroups given by their vector (n,m)-presentations. For some vector (n,m)-presentations, the obtained good combinatorial descriptions imply word problem solvability. The aim of this paper is to give an overview of these results.

VECTOR PRESENTATIONS OF (n,m)-SEMIGROUPS. REDUCTIONS

The following definition was originally given in [4] and improved in [34].

Definition 3.1 [34] For an (n,m)-presentation $\langle B; \Delta\rangle$ of an (n,m)-semigroup, we say that it is a vector (n,m)-presentation of an (n,m)-semigroup, in short vector (n,m)-presentation, and that Δ is a set of vector (n,m)-relations, if the following conditions are satisfied:

(1) if $(i,\mathbf{x})\ \Delta\ (j,\mathbf{y})$, then $i = j$ and $\mathbf{x}, \mathbf{y} \in B^{m,k}$;

(2) if $(i,\mathbf{x})\ \Delta\ (i,\mathbf{y})$, then $(j,\mathbf{x})\ \Delta\ (j,\mathbf{y})$ for every $j \in \mathbb{N}_m$;

(3) if $(i,\mathbf{x})\ \Delta\ b$ for some $b \in B$, then $\mathbf{x} \in B^{m,k}$ and there is $b_1^m \in B^m$, such that $b_i = b$ and for each $j \in \mathbb{N}_m$, $(j,\mathbf{x})\ \Delta\ b_j$;

(4) if $b\ \Delta\ (i,\mathbf{x})$ for some $b \in B$, then $\mathbf{x} \in B^{m,k}$ and there is $b_1^m \in B^m$, such that $b_i = b$ and for each $j \in \mathbb{N}_m$, $b_j\ \Delta\ (j,\mathbf{x})$; and

(5) $\Delta \cap B \times B = \emptyset$.

In other words, an (n,m)-presentation is a vector (n,m)-presentation if only the (n,m)-operation is used in the defining relations.

Example 3.1. [34] Let $B = \{a, b\}$ and let Δ be the following set:

$\Delta = \{((1, aabb), a), ((2, aabb), b),$
$((1, aaabbb), b), ((2, aaabbb), a),$
$((1, aaa), a), ((2, aaa), a)\}$

The relation from Δ can be written in the form: $[aabb] = ab$, $[aaabbb] = ba$ and $[aaa] =$

aa, and they imply that: $ba = [aaabbb] = [a[aabb]b] = [aabb] = ab$. Thus, in the $(3,2)$-semigroup with the given presentation, a and b have to be identified. The relation $[aaa] = aa$ implies that the $(3,2)$-semigroup whose presentation is $\langle B;\Delta\rangle$ is $(A,[\ ])$ where $A = \{a\}$ and $[aaa] = aa$. Note that we did not use the component operations of the $(3,2)$-operation $[\ \ ]$ for the defining relations. Various examples of vector (n,m)-presentations can be found in [4, 31, 34].

Definition 3.2 [34] For a vector (n,m)-presentation $\langle B;\Delta\rangle$ we define a binary semigroup presentation $\langle B;\Lambda\rangle$, where $\Lambda = \Lambda'\cup\Lambda''\cup\Lambda'''$, $\Lambda'\subseteq B^{m,k}\times B^{m,k}$, $\Lambda''\subseteq B^{m,k}\times B^{m}$, $\Lambda'''\subseteq B^{m}\times B^{m,k}$ and $\Lambda',\Lambda'',\Lambda'''$ are defined by:

(1) $\mathbf{x}\ \Lambda'\ \mathbf{y}$ if and only if $(1,\mathbf{x})\ \Delta\ (1,\mathbf{y})$;

(2) $\mathbf{x}\ \Lambda''\ b_1^m$ if and only if $(j,\mathbf{x})\ \Delta\ b_j$ for each $j\in\mathbb{N}_m$;

(3) $b_1^m\ \Lambda'''\ \mathbf{x}$ if and only if $b_j\ \Delta\ (j,\mathbf{x})$ for each $j\in\mathbb{N}_m$.

We say that Λ is induced by Δ, and $\langle B;\Lambda\rangle$ is induced by $\langle B;\Delta\rangle$.

The binary semigroup presentation induced by the vector $(3,2)$-presentation in Example 3.1 is $\langle a,b\,;aabb = ab, aaabbb = ba, aaa = aa\rangle$. Note that the semigroup with this presentation is not trivial, i.e. has more than one element, while the $(3,2)$-semigroup with the corresponding vector $(3,2)$-presentation in Example 3.1 is trivial, i.e. has only one element.

Proposition 3.1 [34] *The class of vector (n,m)-presentations is equivalent to the class of semigroup presentations $\langle B;\Lambda\rangle$ where $\Lambda\subseteq B^{m,k}\times B^{m,k}\cup B^{m,k}\times B^{m}\cup B^{m}\times B^{m,k}$ i.e. there is a bijection between these two classes.*

We note that the empty set is a set of vector (n,m)-relations, that induces a presentation of a free binary semigroup in which the word problem is solvable. But by no means this implies directly that the word problem for free vector valued semigroups is solvable. Establishing the $1-1$ correspondence above, it seemed more achievable to focus on constructing good reductions for vector (n,m)-presentations $\langle B;\Delta\rangle$ and to explore the circumstances under which it might be possible. A couple of investigations have been made in [31]. Here we give some of the conclusions.

Let $\langle B;\Delta\rangle$ be a vector (n,m)-presentation of an (n,m)-semigroup. Providing that there exists a good reduction φ for its induced binary presentation $\langle B;\Lambda\rangle$, a good reduction ψ for $\langle B;\Delta\rangle$ has been constructed in the following cases:

i) If none of the pairs in Λ has length m i.e. if $\Lambda\subseteq B^{m,k}\times B^{m,k}$;

ii) If $\varphi(b_1^m) = b_1^m$ for all $b_1^m\in B^m$;

iii) If φ reduces the length on B^+.

It is easy to notice that i) and iii) are special cases of ii), however we give them independently, since i) was the first conclusion we have obtained and then realized that analogical construction works for wider classes satisfying ii). These results have their improved versions and will be stated as theorems in the next section. Regarding the condition iii), we have proved that the existence of a reduction φ for $\langle B;\Lambda\rangle$ that reduces the length on B^+, allows a construction of a reduction ψ for $\langle B;\Delta\rangle$ that will reduce the norm on $F(B)$.

Theorem 3.2 [31] *Let $\langle B;\Delta\rangle$ be a vector (n,m)-presentation of an (n,m)-semigroup and let φ be a reduction for its induced binary presentation $\langle B;\Lambda\rangle$ satisfying*

$\varphi(x)\neq x \Longrightarrow |\varphi(x)| < x|,\ x\in B^+$.

Then there exists a good reduction ψ for $\langle B;\Delta\rangle$. □

Further improvements of these results and also new once have been obtained thanks to the suggestion of one of the anonymous referees of the paper [34]: to switch the investigations to the language of abstract rewriting systems, instead in the language of reductions only. Straightforward and clearer proofs have been provided through confluent rewriting systems, instead of using good reductions only. Moreover, additional results and important conclusions have been obtained.

ABSTRACT REWRITING SYSTEMS - IMPROVED RESULTS

An abstract rewriting system (ARS) is the most general notion about specifying a set of objects and rules that can be applied to transform them. An ARS is a set A, whose elements are usually called objects, together with a binary relation on A, traditionally denoted by $\longrightarrow$, and called the reduction relation or rewrite relation (rule). Detailed definitions and properties of (abstract) rewriting systems can be found, for example, in [18, 19]. We note that, if a reduction is obtained from a confluent, terminating abstract rewriting system (ARS), then it is a good reduction, and moreover the reduced represent for u is the unique normal form (UNF) for u. (For more details see [18, 19, 34]).

In the construction of a canonical form of a free (n,m)-semigroup (originally given in [7] and introduced above), the good reduction ψ_0 for $\langle B;\emptyset\rangle$ was explicitly defined by induction on the norm. In terms of ARS, the reduction rule corresponding to the map ψ_0 is given by:

$$(j, \mathbf{x}(1,\mathbf{y})(2,\mathbf{y})\dots(m,\mathbf{y})\mathbf{z}) \longrightarrow (j, \mathbf{xyz})$$

With this reduction (or rewrite) rule for $F(B)$, the proof in [7], adjusted to the language of ARS, shows that the ARS obtained by the above rule is confluent and terminating. Hence, it is canonical (also called "complete" or "uniquely terminating"), and the UNF for any $u \in F(B)$ is $\psi_0(u)$. Here we give the results obtained in [34], by the language of ARS.

Theorem 4.1 [34, 31] *Let $\langle B;\Delta\rangle$ be a vector (n,m)-presentation and let $\langle B;\Lambda\rangle$ be its induced semigroup presentation.*

(a) If there is a reduction φ for $\langle B;\Lambda\rangle$ satisfying the condition

(4.1) $\varphi(b_1^m) = b_1^m$ *for all* $b_1^m \in B^m$,

then there is a reduction ψ for $\langle B;\Delta\rangle$ and $(\psi(F(B)), g)$ where

$$g(u_1^n) = \psi(1,u_1^n)\psi(2,u_1^n)\dots\psi(m,u_1^n),$$

is the (n,m)-semigroup $\boldsymbol{F(B)}/\overline{\Delta}$. Moreover, if the reduction φ for $\langle B;\Lambda\rangle$ is good, then the reduction ψ for $\langle B;\Delta\rangle$ is good and the word problem for $\langle B;\Delta\rangle$ is solvable.

(b) If there is a canonical ARS $\mathbf{C}$ *for $\langle B;\Lambda\rangle$, such that the UNF of b_1^m is b_1^m for every $b_1^m \in B^m$, then there is a canonical ARS* $\mathbf{D}$ *for $\langle B;\Delta\rangle$, and the word problem for $\langle B;\Delta\rangle$ is solvable.* □

Remark. The canonical ARS $\mathbf{D}$ for $\langle B;\Delta\rangle$ in the proof of Theorem 4.1 is obtained by the rewriting rules of $\mathbf{C}$ and the rewriting rules of an ARS $\mathbf{A}$ we have constructed, that is terminating and (locally) confluent (Church Rosser), and thus canonical (by Newman Lemma). But all this modulo the UNF for $\mathbf{x}$ obtained by $\mathbf{C}$, since in the rewriting rules, UNF for $\mathbf{x}$ is a black box. (For more details see [34, 18, 21]).

The next result, although a corollary of Theorem 4.1 is stated as a Theorem, because: it can be proven independently of Theorem 4.1, by using a simpler ARS; it was the first step toward the proof of Theorem 4.1; and it is easier to check if it can be applied to a given vector (n,m)-presentation.

Theorem 4.2 [34, 31] *Let $\langle B;\Delta\rangle$ be a vector (n,m)-presentation and $\langle B;\Lambda\rangle$ its induced semigroup presentation. If $\Lambda \subseteq B^{m,k} \times B^{m,k}$, then any reduction φ for $\langle B;\Lambda\rangle$, generates a reduction ψ for the vector (n,m)-presentation $\langle B;\Delta\rangle$ and $(\psi(F(B)), g)$ where*

$$g(u_1^n) = \psi(1,u_1^n)\psi(2,u_1^n)\dots\psi(m,u_1^n),$$

is the (n,m)-semigroup $\boldsymbol{F(B)}/\overline{\Delta}$.

Moreover:

a) if there is a good reduction φ for $\langle B;\Lambda\rangle$, then there is a good reduction ψ for $\langle B;\Delta\rangle$ and the word problem for $\langle B;\Delta\rangle$ is solvable;

b) if there is a canonical ARS for $\langle B;\Lambda\rangle$, then there is a canonical ARS for $\langle B;\Delta\rangle$ and the word problem for $\langle B;\Delta\rangle$ is solvable. □

The following theorem is an improvement of Theorem 4.1.

Theorem 4.3 [34] *Let $\langle B;\Delta\rangle$ be a vector (n,m)-presentation and let φ be a reduction for the induced semigroup presentation $\langle B;\Lambda\rangle$ such that its restriction to B^m is injective. Then, the reduction φ for $\langle B;\Lambda\rangle$, generates a reduction ψ for the vector (n,m)-presentation $\langle B;\Delta\rangle$ and $(\psi(F(B)), g)$ where*

$$g(u_1^n) = \psi(1,u_1^n)\psi(2,u_1^n)\dots\psi(m,u_1^n),$$

is the (n,m)-semigroup $\boldsymbol{F(B)}/\overline{\Delta}$.

Moreover, if B is finite and the reduction φ for $\langle B;\Lambda\rangle$ is good, then the reduction ψ for $\langle B;\Delta\rangle$ is good and the word problem for $\langle B;\Delta\rangle$ is solvable. □

In the following result we have improved Theorem 4.1 - for finite generating sets, obtaining that when B is finite, no extra conditions on the good reduction φ are required.

Theorem 4.4 [34] *Let $\langle B;\Delta\rangle$ be a vector (n,m)-presentation and let B be a finite set. Then, the existence of a good reduction for the induced semigroup presentation $\langle B;\Lambda\rangle$ implies existence of a good reduction for the vector (n,m)-presentation $\langle B;\Delta\rangle$, and the word problem for $\langle B;\Delta\rangle$ is solvable.* □

OPEN PROBLEMS

We still do not have answers to the following questions.

1) Does the existence of good reduction for the induced semigroup presentation $\langle B;\Lambda\rangle$ whose restriction to B^m is injective imply existence of a good reduction for $\langle B;\Delta\rangle$?

The answer is YES for B finite (Theorem 4.3), but we expect that the answer is NO when the set B is not finite. However, future investigations can be made for finding some special classes of vector (n, m)-presentations $\langle B; \Delta\rangle$ where B is not finite, but Theorem 4.3 still holds.

2) Does the existence of a canonical ARS for the induced semigroup presentation $\langle B; \Lambda\rangle$ such that the unique normal forms of different elements from B^m are different, imply existence of a canonical ARS for $\langle B; \Delta\rangle$?

We expect that the answer is NO in this case, even when B is finite. This conclusions shall be proved and also, some special cases that fulfill YES as an answer might be searched.

3) The question if the construction in Theorem 4.1 (or some modified version) is possible for vector (n, m)-presentations of (n, m)-semigroups $\langle B; \Delta\rangle$ not satisfying the condition (4.1), remains open. Some of the problems that arise here are that some elements from B have to be identified in the (n, m)-semigroup, although they are different in the semigroup $\langle B; \Lambda\rangle$, or some elements of the form $(i, \mathbf{x})$ and $(j, \mathbf{y})$ for $i \neq j$ have to be identified.

The discussion above leads to the following question.

4) Is construction of a good reduction (or canonical ARS) possible in general case, i.e. for vector (n, m)-presentations $\langle B; \Delta\rangle$ permitting the corresponding set of induced binary relations Λ to contain pairs with length m?

The general answer is most probably NO (some counter examples might be found). Perhaps further classifications on Δ i.e. Λ shall be made, which would lead to appropriate conclusions and/or guides for future investigations.

Summing up, the existence of a good combinatorial description (or a canonical ARS) for $\langle B; \Lambda\rangle$ not necessarily implies existence of a good combinatorial description for $\langle B; \Delta\rangle$. Theorem 4.4 indicates that the above is true when B is finite. However,

5) Does the solution of the word problem for $\langle B; \Lambda\rangle$ imply solution to the word problem for $\langle B; \Delta\rangle$?

At this moment we do not have an answer to this question, although we have some indication that, in general, the answer is NO for B infinite and is YES for B finite.

APPLICATIONS IN VARIETIES OF (n, m)-SEMIGROUPS; GENERALIZATIONS THAT INCORPORATE NEW IDEAS

The introduction of vector (n, m)-presentations of (n, m)-semigroups has led to noticeable results for varieties of (n, m)-semigroups. The definition of a variety of (n, m)-semigroups was originally given in [2]. Vector varieties of (n, m)-semigroups and vector (n, m)-presentations in such varieties were introduced in [4]. Recent investigations were made in [35, 36, 37]. In [35] a direct description of the complete system of (n, m)-identities for a variety of (n, m)-semigroups is obtained. In [36] a characterization of vector varieties of (n, m)-semigroups is made. It is shown that the class of vector varieties of (n, m)-semigroups is a proper subset of the class of varieties of (n, m)-semigroups (when $m \geq 2$), and necessary and sufficient conditions for a variety of (n, m)-semigroups to be a vector variety are provided. In [37] a direct proof of Birkhoff's HSP theorem for varieties of (n, m)-semigroups is given. Moreover, a corresponding analog of this theorem for vector varieties of (n, m)-semigroups (when $m \geq 2$) is obtained.

The results exposed in this paper can be applied for appropriate classes of vector varieties of (n, m)-semigroups. Consequently, good reductions for vector (n, m)-presentations in some classes of vector varieties of (n, m)-semigroups might be constructed. This would lead to an existence of good descriptions for free objects in such (n, m)-varieties. There are various open questions concerning (vector) varieties of (n, m)-semigroups and numerous investigation possibilities within.

Vector valued semigroups provide a way of obtaining new languages. If we think of a binary operation as a process that from two information produces one information, then we can think of an $(m+k, m)$-operation as a process that from $m + k$ information, produces m information. Several authors (D. Dimovski, V. Manevska) have investigated formal vector valued languages and automata [15, 20]. The aim of our work, in a way, is to obtain a better understanding of these complicated languages. Afterwards, they might find possible application in ICT security systems. Quantum computers would additionally support this idea, and hopefully open new opportunities for incorporating these formal languages within ICT security systems. It is quite possible that the development of the combinatorial theory of (n,m)-semigroups has a bright future in front.

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ЗА РЕДУКЦИИ НА ПРЕТСТАВУВАЊА НА ВЕКТОРСКО ВРЕДНОСНИ ПОЛУГРУПИ: ПРЕГЛЕД И ОТВОРЕНИ ПРОБЛЕМИ

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Посветено на нашиот сакан професор Ѓорѓи Чупона

Наоѓањето на комбинаторен опис за (n, m)-полугрупа зададена со (n, m)-претставување $\langle B; \Delta \rangle$ е прилично тешка задача и комплексен проблем. Поголемиот број добиени резултати се однесуваат на специјални класи претставувања на (n, m)-полугрупи наречени векторски (n, m)-претставувања на (n, m)-полугрупи. Истите индуцираат соодветни претставувања на бинарни полугрупи, поради што прашањето за постоење на добар комбинаторен опис на $\langle B; \Delta \rangle$ е тесно поврзано со прашањето за постоење на добар комбинаторен опис на соодветната индуцирана бинарна полугрупа $\langle B; \Lambda \rangle$. Правиме прегед на овие резултати, при што ги класифицираме условите под кои постоењето на добар комбинаторен опис за $\langle B; \Lambda \rangle$ имплицира решливост на проблемот на зборови во $\langle B; \Delta \rangle$. Дефинираме неколку отворени проблеми, посочуваме примена на добиените резултати во многуобразија (n, m)-полугрупи и даваме насоки за идни истражувања.

Клучни зборови: (n, m)-полугрупа, (n, m)-претставување, редукција, проблем на зборови