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МАКЕДОНСКА АКАДЕМИЈА НА НАУКИТЕ И УМЕТНОСТИТЕ

ОДДЕЛЕНИЕ ЗА ПРИРОДНО-МАТЕМАТИЧКИ И БИОТЕХНИЧКИ НАУКИ

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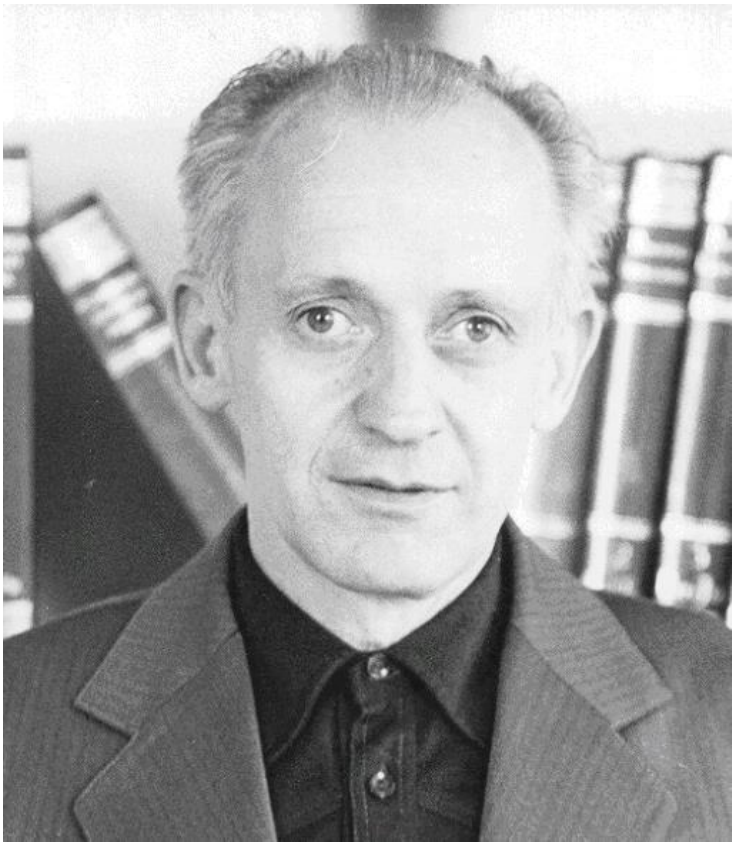

АКАДЕМИК ЃОРЃИ ЧУПОНА
(По повод 90 години од неговото раѓање)

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Академик Ѓорѓи Чупона целиот свој живот и творечка енергија ги има посветено на математиката. Неговите постигнувања се огромни. Така, не е едноставно да се напише текст од неколку страници кој ќе даде доволно добра слика за тие постигнувања. Со огромно задоволство, како негов студент, соработник и долго време негов многу близок пријател, го пишувам овој текст. Ќе се потрудам да бидам доволно прецизен и концизен, каков што беше мојот професор Чупона.

Академик Ѓорѓи Чупона е роден на 10 април 1930 година во пелистерското село Маловиште. Основно и средно образование оформил во Битола, а дипломирал во 1953 година на Филозофскиот факултет во Скопје, група математика. Докторирал во 1959 година на Природно-математичкиот факултет (ПМФ) во Скопје, на тема „Прилог кон теоријата на алгебарските структури", под менторство на професор д-р Владимир Девиде од Загреб. Кус период работи како наставник во средно училиште до неговиот избор за асистент по математика на Филозофскиот факултет во 1955 година. Во 1961 година е избран за доцент на Техничкиот (подоцна Електро-машински) факултет во Скопје. За вонреден професор е избран во 1967 година, кога се вратил на ПМФ, каде што во 1972 година е избран за редовен професор по алгебра. Од 1985 година до пензионирањето во 1994 година, беше член на Институтот за информатика на ПМФ.

Во текот на 1964 година бил на специјализација на Универзитетот во Манчестер, Велика Британија.

Во 1979 година е избран за дописен, а во 1983 година за редовен член на Македонската академија на науките и уметностите (МАНУ).

Има добиено повеќе признанија и награди за покажаните резултати, како што се: плакета на Универзитетот „Кирил и Методиј“, наградата „11 Октомври“ во 1966 година, Орден за заслуги на народот со сребрен венец и Орден на трудот со златен венец.

Во македонската математика академик Чупона е основоположник на алгебрата, како и на дисциплините тесно поврзани со неа, од сите нејзини аспекти. Може слободно да се каже дека целиот научен кадар по алгебра во Македонија, а и кадар надвор од Македонија, е развиен под негово суштинско раководство, иако бил ментор на само четворица докторанди: Наум Целакоски, Билјана Јанева, Жанета Попеска (тројцата од Македонија) и на Биљана Зековиќ (од Црна Гора). Исто така, тој имал поголем или помал удел во развојот на голем број математичари, информатичари и истражувачи од други области во Македонија и надвор од неа. Енергичен, упорен и трпелив, Чупона собираше околу себе млади заинтересирани студенти и соработници, постојано ги поттикнуваше, водеше и работеше со нив до добивање соодветни резултати.

Според постигнатите научни резултати, тој успеа да ја направи македонската алгебра препознатлива и во светски рамки. Соработката остварена со редица математичари од поранешните југословенски и балкански простори (ќе наведам само неколкумина: Владимир Девиде, Славиша Прешиќ, Светозар Милиќ, Јанез Ушан, Калчо Тодоров, Валентин Данилович Белоусов), како и големиот број печатени трудови го истакнаа академик Чупона како еден од водечките алгебристи во СФР Југославија, а и пошироко. Беше иницијатор за организирањето на следниве конференции што се одржуваа на просторите од поранешна Југославија:

I алгебарска конференција, Скопје, 1980;

II алгебарска конференција, Нови Сад, 1981;

III алгебарска конференција, Белград, 1982;

IV конференција алгебра и логика, Загреб, 1984;

V конференција алгебра и логика, Цетиње, 1986;

VI конференција алгебра и логика, Сараево, 1987;

VII конференција алгебра и логика, Марибор, 1989;

VIII конференција алгебра и логика, Нови Сад, 1998.

Во Скопје во 1982 година тој го организираше Симпозиумот „n-арни структури“, а следната година беше организиран Втор меѓународен симпозиум „n-арни структури“ во Варна, Бугарија.

Во текот на неговиот работен век беше раководител на голем број научни проекти, финансирани од СИЗ за научни дејности и од МАНУ. Во рамките на тие проекти, редовно организираше семинари, кои одиграа значајна улога во афирмирањето и во научното издигнување на голем број негови ученици, а подоцна и соработници. Некои од семинарите беа одржувани како еднонеделни научни собири во други места од Републиката (Охрид, Струга и Пелистер) со учество на математичари од Македонија и повеќе универзитетски центри надвор од Македонија. Од работата на тие проекти се оформени и објавени следниве монографии во кои академик Чупона има големо учество:

- *Зборник на трудови од Алебарската конференција*, Скопје (1980),152 стр.;
- *Зборник на трудови од Симпозиумот n-арни структури,* МАНУ, Скопје (1982), 289 стр.;
- *Векторско вредносни полугрупи и групи,* МАНУ, Скопје (1988), 198 стр.;
- *Комплексни комутативни веторско вредносни групи,* МАНУ, Скопје (1992), 120 стр.;
- *Научни трудови по алгебра во Република Македонија* 1950 – 1980, Том 1, МАНУ, Скопје (2008), 730 стр.;
- *Научни трудови по алгебра во Република Македонија* 1981 – 1990, Том 2, МАНУ, Скопје (2008), 704 стр.;
- *Научни трудови по алгебра во Република Македонија* 1991 – 2007, Том 3, МАНУ, Скопје (2008), 822 стр.

Огромна е неговата улога во развојот на информатиката во Македонија. Беше еден од иницијаторите за формирањето на Математичкиот институт со нумерички центар. Советот на Универзитетот во Скопје (подоцна Универзитет „Кирил и Методиј“) на 21 јуни 1966 година, донесе одлука за основање Математички институт со нумерички центар на Универзитетот во Скопје (МИНЦ) во која: 1) се наведени неговите задачи; 2) за вршител на должноста директор се именува д-р Благој Попов, професор на ПМФ; и 3) се образува матична комисија во состав: Јоже

Улчар од ПМФ, д-р Милорад Радоњиќ од Земјоделскиот факултет, д-р Ѓорѓи Чупона од Електро-машинскиот факултет, Димитар Битраков од Архитектонско-градежниот факултет, Вања Хаџиев од Економскиот институт и д-р Исак Таџер, проректор. Може слободно да се каже дека академик Чупона го посвети целиот свој работен век со несмален елан на исполнување на задачите од одлуката за основање на МИНЦ. Ќе цитирам четири од нив во оригинална форма, кои Чупона целосно ги спроведуваше и кои најдобро ќе го осветлат неговото животно дело:

- *да ја организира и усмерува научно-истражувачката работа по незастапените математички дисциплини, преку директно обработување на одделни проблеми или преку изучување на одделни области;*
- *да дава помош на стручњаците при научно истражувачката работа од областа на математиката, техничките, економските и др. науки;*
- *да издава математички списанија на Универзитетот, во кои да ги објавува и непосредно соопштува резултатите од својата научно-истражувачка работа и да ја популаризира математиката;*
- *да допринесува за подобрувањето на учебниците, скриптите и другите помагала за високошколската и средношколската настава по математика.*

Од 1969 до 1973 година, академик Чупона беше и директор на МИНЦ. Со реформата на Универзитетот во Скопје во 1977 година, МИНЦ се вклопи во новоформираниот Математички факултет. Во 1985 година, Универзитетот се реформира повторно и се врати на формата од пред 1977 година, но МИНЦ не беше обновен. Се сеќавам дека во тој период Чупона ме замоли да подготвам материјал за формирање центар во МАНУ по урнекот на МИНЦ, нагласувајќи ги неговите задачи. Затоа, според мое мислење, повеќето од задачите во претходно споменатата одлука за основањето на МИНЦ се пишувани од Чупона по предлог на професор Благој Попов. Бидејќи таков центар, во тој момент, не се формира во МАНУ, академик Чупона успеа, со огромни заложби, да се формира посебен институт на новоформираниот ПМФ, Институтот за информатика, каде што работеше до неговото пензионирање.

Во склоп на неговиот интерес за откривање и работа со талентирани ученици за математика, во склоп на МИНЦ, академик Чупона иницира организирање математички школи за учениците од средните училишта, на кои предавачи беа еминентни математичари и информатичари. Мојата прва средба со него беше на Летната математичка школа во Охрид во 1972 година. Посветеноста на академик Чупона за воведување и изучување нови математички и информатички области може да се види и од темите обработувани на математичките школи. Ќе наведам некои од нив:

- Булова алгебра и нејзина примена;
- Елементи од теоријата на веројатноста;
- Програмирање на електронски пресметувачки машини;
- Комбинаторика;
- Елементи од аналитичка геометрија;
- Фортран програмирање;
- Елементи од теоријата на множества и топологијата;
- Квантна механика;
- Постова и Тјурингова машина и нормални алгоритми;
- Математичка логика;
- Основи на линеарно програмирање;
- Основи на теорија на графови и примена;
- Алгоритми, сметачки машини и примена;
- Полугрупи и конечни автомати;
- Мрежно планирање;
- Диференцни равенки;
- Конечни автомати и регуларни јазици;
- Линеарни модели и оптимизација;
- Топологија преку логика;
- Динамички системи и хаос.

Голем број учесници на овие летни школи, речиси од сите места во Македонија, подоцна станаа врвни научници не само во областа на математиката, туку и речиси во сите други научни области. Ќе споменам само тројца од нив, кои се иста возраст со мене: Љупчо Коцарев (електроинженер и доктор по физика), Милан Ќосевски (доктор на машински науки) и Миле Крајчевски (доктор на математички науки). Почнувајќи од нашите средношколски денови, преку летните школи, сите имаме плодна соработка до денес. Во тоа време, Чупона беше љубител на играта со карти белот, која е многу популарна во Битола. Бидејќи сум од Битола, на тие летни школи, со Чупона игравме белот и ги учевме учесниците од другите места да ја играат таа игра.

Улогата што ја одиграа учебниците „Предавања по алгебра“, Книга I, „Предавања по алгебра“, Книга II, „Алгебарски структури и реални броеви“, како и „Предавања по виша математика“ I, II, III и „Виша математика“ I, II, III, IV, наменети за студентите од техничките факултети, кои имаат повеќе изданија, чиј автор, односно коавтор е академик Чупона, е огромна. Овие учебници имаа и сè уште имаат незаменливо влијание во издигањето на алгебарската и, пошироко, математичката култура во Македонија.

За да ја илустрирам грижата на Чупона за добро образование по математика, ќе споменам дека книгата „Алгебарски структури и реални броеви“, е пишувана со цел средношколски наставници по математика да го осовременат своето математичко знаење. Во тоа време, Жанета Попеска и јас, како студенти, на сугестија од нашиот професор Чупона, ја препишавме конечната верзија на книгата, решавајќи ги сите задачи, што ни помогна многу да научиме.

Научната работа на академик Чупона е целосно посветена на изучувањето на алгебарските структури. Започнува во педесеттите години од минатиот век во периодот на зголемениот интерес за универзалната алгебра, којашто потоа стана дел од бурниот развој на алгебарските n-арни структури. Неговото дело опфаќа широка листа проблеми од голем број подрачја на алгебрата. Трудовите се одликуваат со прецизност и концизност во формулацијата и доказите на теоремите, со оригиналост, длабочина и општост во расудувањата.

Според третираната проблематика, опусот на научната работа на професорот Чупона може да се распореди во следниве неколку групи, при што броевите се однесуваат на референции од неговата библиографија.

1. *Класични алгебарски структури:* (1, 3, 4, 8, 13, 14, 15, 16, 25, 52).

2. *Релации и операции:* (2, 5, 6, 7, 9, 10, 17, 21, 53).

3. *Алгебарски структури со асоцијативни n-арни операции:* (11, 12, 18, 19, 23, 24, 32, 33, 38, 39, 45).

4. *Сместување на алгебри во полугрупи:* (22, 27, 29, 30, 36, 40, 49, 55, 57, 73).

5. *Алгебри и обопштени подалгебри:* (20, 28, 31, 34, 35, 37, 43, 46, 47, 48, 50, 51, 54, 58, 69, 71, 72, 75, 76, 77, 78, 81).

6. *Векторско вредносни алгебарски структури:* (26, 41, 42, 44, 56, 59, 60, 61, 62, 63, 64, 65, 66, 67,68, 70, 74, 87, 89, 90).

7. *Слободни објекти во многуобразија групоиди:* (79, 80, 82, 83, 84, 85, 86, 88, 91, 92, 93, 94, 95, 96).

Ќе дадам краток опис на секоја од овие седум групи.

Трудовите од групата **1** третираат проблеми во врска со конечни полиња, диедрални групи, редуцибилни полугрупи, квазипрстени и специјална класа компатибилни полугрупи. Даден е нов опис на комплетно простите полугрупи.

Резултатите од групата **2** се за релации меѓу операции. Некои од нив се бинарни релации меѓу финитарни операции коишто се обопштувања на некоја од релациите комутативност, асоцијативност или дистрибутивност. Во n-арниот случај ($n \geq 3$), докажано е дека од делумна асоцијативност и постоење неутрален елемент или некој друг услов, следува целосна асоцијативност. Испитани се ω-арни асоцијативни операции, каде што ω е пребројлив бесконечен ординален број и докажано е дека не постојат ω-групи со повеќе од еден елемент.

Трудовите во групата **3** се за алгебарските структури со една n-арна асоцијативна операција ($\mathrm{n} \geq 3$), наречени n-полугрупи или алгебарски структури со неколку n-арни асоцијативни операции, наречени асоцијативи. Покажано е дека секоја n-полугрупа е n-потполугрупа од полугрупа. Овој резултат (наречен теорема на Пост за n-полугрупи) овозможува разни својства на n-полугрупите да се испитуваат со соодветни својства на полугрупите. Дадени се описи на асоцијативи, при што посебно внимание е посветено на асоцијативите што може да се сместат во полугрупа. Воведен е поимот за $[\mathrm{n}, \mathrm{m}]$-прстен и е докажано дека секој $[\mathrm{n}, \mathrm{m}]$-прстен може да биде сместен во прстен. Дадени се карактеризации на n-потполугрупи на некои класи полугрупи (на пример, периодични и комутативни полугрупи).

Во трудовите од групата **4** се добиени неколку значајни резултати за репрезентација, односно сместување на алгебри во полугрупи. Дадени се сместувања на неколку класи алгебри во полугрупи поврзани со теоремата на Кон-Ребане. Покажано е дека секоја тополошка алгебра може да биде вметната во тополошка полугрупа во смисла на торемата на Кон-Ребане и дека универзалната покривка на тополошка n-група е тополошка група. Опишана е класата алгебри кои се подалгебри на полумрежи. Докажано е дека секоја полугрупа од некои класи нормирани полугрупи може да биде сместена во полугрупа од операции.

Групата **5** се однесува на проблеми во врска со обопштени подалгебри, делумни подалгебри, многуобразија алгебри и унарни алгебри. Прашања за сместување на алгебри во полугрупи се испитуваат во претходните две групи, додека во оваа група ваквите прашања се разгледуваат во поопшта форма. Воведени се сместувачки алгебри и дадена е нивна репрезентација како алгебри на операции. Опишана е класа на делумни алгебри што е обопштување на класата сместувачки алгебри. Добиен е доволен услов за многуобразие да биде n-многуобразие. Карактеризирани се полиномни подалгебри од алгебрите од некои многуобразија алгебри. Докажани се резултати за подалгебрите, хомоморфизмите и слободните објекти во некои класи полиалгебри и за класата хиперполугрупи. Испитувана е посебна класа на идентитети во n-групоиди, наречени примитивни n-идентитети и е докажано дека проблемот на зборови е решлив во секое примитивно n-многуобразие.

Во групата **6** се воведува и испитува поимот за векторско вредносни алгебарски структури. Вистинското истражување на овие структури започнува со трудот *Векторско вредносни полугрупи* (56), каде што е докажано дека секоја (n,m)-полугрупа може да биде покриена со полугрупа. Речиси сите резултати од полиадични полугрупи поврзани со теоремата на Пост се пренесуваат и за (n,m)-полугрупите, ама тоа не е случај кога се во прашање резултати од егзистенцијална природа. Истото ова важи и за векторско вредносните групи. Докажани се векторско вредносни аналогии на теоремите на Кон-Ребане и Пост. Карактеризирани се слободните векторско вредносни групоиди, полугрупи и групи. Докажано е дека постои аналогија помеѓу теоријата на групи и теоријата на (2m,m)-групи. Воведен е поимот за инјективни векторско вредносни полугрупи и е докажано дека класата од слободни векторско вредносни полугрупи е вистинска поткласа од класата инјективни векторско вредносни полугрупи. Развиена е комбинаторна теорија за векторско вредносни полугрупи.

Трудовите во групата **7** се однесуваат на многуобразија групоиди дефинирани со некои идентитети. Во многу од овие трудови е даден каноничен опис на слободните групоиди и карактеризирани се нивните подгрупоиди. За некои од овие многуобразија е докажано дека проблемот на зборови е решлив. Испитувањето на многуобразието групоиди дефинирано со идентитетот $x^n = x$ доведе до поим за групоиден степен. За некои од овие многуобразија групоиди е докажана таканаречената теорема на Брак, односно е докажано дека слободни групоиди во многуобразието се карактеризирани со инјективните групоиди во истото многуобразие.

На крајот ќе кажам дека академик Чупона подеднакво се однесуваше кон сите, без оглед на нација, раса и религија. Во срцето имаше љубов за секого. Беше неизмерно скромен и совесен човек, но и човек кој знаеше да ни укаже на грешките. Имав голема привилегија да бидам негов ученик и соработник за што сум му бескрајно благодарен. Денес многу мои постапки наликуваат на неговите.

Академик Ѓорѓи Чупона почина на 16 декември 2009 година, а тоа, всушност, има голема симболика. На 16 декември 1946 година почнал со работа Филозовскиот факултет, од кој подоцна израсна и ПМФ – неговиот втор дом.

ACADEMICIAN GJORGJI ČUPONA
(On the occasion of 90 years since his birth)

Dončo Dimovski

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Academician Gjorgji Čupona has dedicated his entire life and creative energy to mathematics. His achievements are huge. Thus, it is not easy to write several pages of text that will give a good enough picture of those achievements. With a great pleasure, as his student, collaborator and longtime his very close friend, I am writing this text. I will do my best to be precise and concise enough, as my Professor Čupona was.

Academician Gjorgji Čupona was born on April 10, 1930 in the Pelister village Malovishte. He completed his primary and secondary education in Bitola, and graduated in 1953 at the Faculty of Philosophy in Skopje, group for mathematics. He received his PhD degree in 1959 at the Faculty of Natural Sciences and Mathematics (FNSM) in Skopje, on the topic "Contribution to the theory of algebraic structures", under the supervision of Prof. Dr. Vladimir

Devide from Zagreb. For a short time he worked as a high school teacher until his election as a mathematics assistant at the Faculty of Philosophy in 1955. In 1961 he was elected for an assistant professor at the Technical Faculty (later Faculty of Electrical and Mechanical Engineering) in Skopje. He was promoted to an associate professor in 1967, when he returned to FNSM, where in 1972 he was promoted to a professor of algebra. From 1985 until his retirement in 1994, he was a member of the Institute of Informatics at FNSM.

During 1964 he was on a specialization at the University of Manchester, UK.

In 1979 he was elected a corresponding, and in 1983 a full member of the Macedonian Academy of Sciences and Arts (MASA).

He has received several recognitions and awards for his achievements, such as: plaque of the University "Cyril and Methodius", the award "October 11" in 1966, the Order of merit for the People with a silver wreath and the Order of labor with a golden wreath.

In Macedonian mathematics, Academician Čupona is the founder of algebra, as well as the disciplines closely related to it, in all of its aspects. It can be freely said that the entire algebra scientific staff in Macedonia, as well as staff outside Macedonia, was developed under his substantial leadership, although he was a supervisor to only four doctorates: Naum Celakoski, Biljana Janeva and Žaneta Popeska (from Macedonia) and Biljana Zeković (from Montenegro). He also had a greater or lesser share in the development of many mathematicians, computer scientists and researchers from other fields inside and outside Macedonia. Energetic, persistent and patient, Academician Čupona gathered around himself young interested students and collaborators, constantly encouraging, guiding and working with them, until appropriate results were obtained.

According to the achieved scientific results, he managed to make the Macedonian algebra recognizable in the world. The cooperation achieved with a number of mathematicians from the former Yugoslavia and the Balkans (I will mention only a few; Vladimir Devide, Slaviša Prešić, Svetozar Milić, Janez Ušan, Kalčo Todorov, Valentin Danilovič Belousov), as well as the large number of published papers highlighted Academician Čupona as one of the leading algebraists in SFR Yugoslavia and beyond. He is the initiator for the organization of the following conferences that were held in the former Yugoslavia:

I Algebraic Conference, Skopje, 1980;

II Algebraic Conference, Novi Sad, 1981;

III Algebraic Conference, Belgrade, 1982;

IV Conference Algebra and Logic, Zagreb, 1984;

V Conference Algebra and Logic, Cetinje, 1986;

VI Conference Algebra and Logic, Saraevo, 1987;

VII Conference Algebra and Logic, Maribor, 1989;

VIII Conference Algebra and Logic, Novi Sad, 1998.

In Skopje in 1982 he organized the Symposium "n-ary Structures" and the following year the Second International Symposium " n-ary Structures" was organized in Varna, Bulgaria.

During his working life he was the manager of a number of scientific projects, funded by SIZ for scientific activities and MASA. Within those projects he regularly organized seminars, which played a significant role in the affirmation and scientific advancement of a large number of his students and later collaborators. Some of the seminars were held as one-week scientific gatherings in other places in the Republic (Ohrid, Struga and Pelister) with a participation of mathematicians from Macedonia and several university centers outside Macedonia. From the work of those projects, the following monographs were created and published, in which Academician Čupona has substantial participation:

- *Proceedings of the Algebraic conference*, Skopje, (1980), 152 p.;
- *Proceedings of the Symposium n-ary Structeres*, MASA, Skopje, (1982), 289 p.;
- *Vector valued semigroups and groups*, MASA, Skopje, (1988), 198 p.;
- *Complex commutative vector valued groups*, MASA, Skopje, (1992), 120 p.;
- *Scientific papers on algebra in the Republic of Macedonia* 1950 – 1980, Vol. I, MASA, Skopje, (2008), 703 p.;
- *Scientific papers on algebra in the Republic of Macedonia* 1981 – 1990, Vol. II, MASA, Skopje, (2008), 704 p.;
- *Scientific papers on algebra in the Republic of Macedonia* 1991 – 2007, Vol. III, MASA, Skopje, (2008), 822 p.

Čupona's role in the development of computer science in Macedonia is huge. He was one of the initiators for the establishment of the Mathematical Institute with a numerical center. The Council of the University of Skopje (later "Cyril and Methodius University") on June 21, 1966 adopted a document for the establishment of the Mathematical Institute with a numerical center at the University of Skopje (MINC) in which: 1) a lists of its tasks is given; 2) Dr. Blagoj Popov, professor at FNSM, is appointed acting director; and 3) a parent commission is formed consisting of: Jože Ulčar from the FNSM, Dr. Milorad Radonjić from the Faculty of Agriculture, Dr. Gjorgji Čupona from the Faculty of Electrical and Mechanical Engineering, Dimitar Bitrakov from the Faculty of Architecture and Civil Engineering, Vanja Hadžiev from the Institute of Economics and Dr. Isak Tadzer, Vice Rector. It is not exaggerating to say that Academician Čupona dedicated his entire working life with undiminished enthusiasm to the fulfillment of the tasks incorporated within the document for the establishment of MINC. I will quote four of them (translated from their original form), which Čupona fully implemented and which will best illuminate his life work.

- *to organize and direct the scientific research work in the unrepresented mathematical disciplines, through direct elaboration of certain problems or through study of certain areas;*
- *to provide assistance to experts in scientific research work in the field of mathematics, technical, economic and other sciences;*
- *to establish mathematical journals at the University for publishing and directly announcing the results of its scientific research work and to popularize mathematics;*
- *to contribute to the improvement of textbooks, scripts and other aids for university and secondary school mathematics teaching*.

From 1969 to 1973, Academician Čupona was also the director of MINC. With the reform of the University of Skopje in 1977, the MINC merged within the newly formed Mathematical Faculty. In 1985 the University was reformed again and returned to its pre-1977 form, but the MINC was not restored. I remember that, in that period, Čupona asked me to prepare material for establishment of a center at MASA, resembling MINC, emphasizing its tasks, so my opinion is that most of the tasks in the aforementioned document for the establishment of MINC, were written by Čupona, on a suggestion by Professor Blagoj Popov. Since such a center was not established at MASA at that moment, Academician Čupona managed, with great efforts, to establish a special institute of the newly established FNSM, the Institute of Informatics, where he worked until his retirement.

As a part of his interest in discovering and working with talented students in mathematics, in the framework of MINC, Academician Čupona initiated the organization of mathematical schools for high school students, taught by eminent mathematicians and computer scientists. My first meeting with him was at the Summer Mathematical School in Ohrid in 1972. The commitment of Academician Čupona for introducing and studying new mathematical and computer science fields can be seen from the topics covered in the mathematical schools. I will list some of them:

- Boolean algebra and its application;
- Elements of probability theory;
- Programming on electronic computing machines;
- Combinatorics;
- Elements of analytical geometry;
- Fortran programming;
- Elements of set theory and topology;
- Quantum mechanics;
- Post and Turing machine and normal algorithms;
- Mathematical logic;
- Basics of linear programming;
- Basics of graph theory and applications;
- Algorithms, computer machines and applications;
- Semigroups and finite automata;
- Network planning;
- Difference equations;
- Finite automata and regular languages;
- Linear models and optimization;
- Topology via logic;
- Dynamical systems and chaos.

Large number of participants at these Summer schools, from almost all places in Macedonia, later became eminent scientists, not only in mathematics, but in almost all of the other sciences. I will mention only three of them, who are the same age as me: Ljupčo Kocarev (Elecrical engeneer and PhD in Physics), Milan Kjosevski (PhD in Mechanical Sciences) and Mile Krajčevski (PhD in Mathematics). Starting from our high school days and these Summer schools, we all have fruitful collaboration till today. In that period, Čupona was a fan of the card game Belote, that is very popular in Bitola. Since I am from Bitola, at these Summer schools, with Čupona we were playing Belote, and we were teaching participants from other places to play this card game.

The role played by the textbooks: "Lectures in Algebra", Book I, "Lectures in Algebra", Book II, "Algebraic Structures and Real Numbers", as well as "Lectures in Calculus", Books I, II, III and "Calculus", Books I, II, III, IV, intended for students from the technical faculties, which have several editions, whose author, i.e. co-author is Academician Čupona is immense. All those textbooks had and still have an irreplaceable impact in the rise of algebraic and, more broadly, mathematical culture in Macedonia.

To illustrate Čupona's care for good education in mathematics, I will mention that the book "Algebraic Structures and Real Numbers" was written mainly for high school teachers of mathematics to update their mathematical knowledge. In that period, Žaneta Popeska and myself, as students, on suggestion by our Professor Čupona, rewrote the final version of the book, by solving all the exercises, that helped us to learn a lot.

The scientific work of Academician Čupona is entirely devoted to the study of algebraic structures. It began in the 1950s during a period of growing interest in universal algebra, which later became part of the turbulent development of algebraic *n*-ary structures. His work covers a wide range of problems in a number of areas of algebra. His papers are characterized by precision and conciseness in the formulation and proofs of the theorems, with originality, depth and generality in reasoning.

According to the topics under consideration, the scientific work of Academician Čupona may be separated into several groups listed below, where the numbers correspond to the references in his Bibliography.

1. *Clasical algebraic structures:* (1, 3, 4, 8, 13, 14, 15, 16, 25, 52).

2. *Relations and operations:* (2, 5, 6, 7, 9, 10, 17, 21, 53).

3. *Algebraic structures with n-ary associative operations:* (11, 12, 18, 19, 23, 24, 32, 33, 38,39, 45).

4. *Representations of algebras in semigroups:* (22, 27, 29, 30, 36, 40, 49, 55, 57, 73).

5. *Algebras and generalized subalgebras:* (20, 28, 31, 34, 35, 37, 43, 46, 47, 48, 50, 51, 54, 58, 69, 71, 72, 75, 76, 77, 78, 81).

6. *Vector valued algebraic structures:* (26, 41, 42, 44, 56, 59, 60, 61, 62, 63, 64, 65, 66, 67,68, 70, 74, 87, 89, 90).

7. *Free objects in varieties of groupoids:* (79, 80, 82, 83, 84, 85, 86, 88, 91, 92, 93, 94, 95, 96).

I will present a short description of each of these seven groups.

The papers in the group **1** consider some problems in finite fields, dihedral groups, reducible semigroups, quasirings, and a special class of compatible semigroups. A new description of comletely simple semigroups is given.

The results in the group **2** are about relations among operations. Some of them are binary relations among finitary opeartions, that are generalizations of some of the relations commutatitivity, associativity or distributivity. In the n-ary case ($n\geq 3$), it is shown that a partial associativity and the existence of a neutral element or some other condition implies the complete associativity. Associative ω-ary operations, where ω is a countable infinite ordinal number, are examined and it is shown that there do not exist ω-groups with more than one element.

The papers in the group **3** are about the algebraic structures with one n-ary associative operation ($n \geq 3$), called n-semigroups, or algebraic structures with several n-ary associative operations, called associatives. It is shown that any n-semigroup is an n-subsemigroup of a semigroup. This result (called Post Theorem for n-semigroups), allows various properties of n-semigroups to be examined with corresponding properties of semigroups. Descriptions of associatives are given, with special attention being paid to associatives that can be embedded into a semigroup. The notion of an [n,m]-ring is introduced and it is shown that any [n,m]-ring can be embedded in a ring. A characterizations of n-subsemigroups of some classes of semigroups, (for example periodic and commutative semigroups) are given.

In the papers from the group **4**, several significant results are obtained for representation, i.e. the embedding of algebras in semigroups. Embeddings of several classes of algebras in semigroups related to Cohn-Rebane's Theorem are given. It is shown that any topological algebra can be embedded into a topological semigroup in the sense of Cohn-Rebane's Theorem and that the universal covering of a topological n-group is a topological group. The class of algebras that are subalgebras of semilattices is described. It is shown that any semigroup from some classes of normed semigroups can be embedded in a semigroup of operations.

The group **5** is concerned with problems related to generalized algebras, partial algebras, varieties of algebras, and unary algebras. Questions about the embedding of algebras in semigroups are examined in the previous two groups, while in this group such questions are considered in a more general form. Insertion algebras are introduced and represented as algebras of operations. A class of partial algebras is described, which is a generalization of the class of insertion algebras. A sufficient condition is obtained for a variety to be an n-variety. Polynomial subalgebras of algebras from some variety of algebras are characterized. Results have been presented for subalgebras, homomorphisms, and free objects in some classes of polyalgebras and for the class of hypersemigroup. A special class of identities in n-groupoids, called primitive n-identities, are examined and it is shown that the word problem is solvable in any primitive n-variety.

In the group **6** the notion of vector valued algebraic structures is introduced and examined. The true research on these structure starts with the paper *Vector valued semigroups* (56), where it is shown that any (n,m)-semigroup can be covered by a semigroup. Almost all results from poliadic semigroups connected with Post's Theorem translate to (n,m)-semigroups, but that is not the case concerning the results of the existencial nature. The same is true for the vector valued groups. Vector valued analogs of Cohn-Rebane and Post Theorems are proven. Free vector valued groupoids, semigroups and groups are characterized. It is shown that there is an analogy between the group theory and the theory of (2m,m)-groups. The notion of an injective vector valued semigroup is introduced and it is shown that the class of free vector valued semigroups is a proper subclass of the class of injective vector valued semigroups. A combinatorial theory of vector valued semigroups is developed.

The papers in the group **7** are concerned with varieties of groupoids defined by some identities, and in majority of them, a canonical description of free groupoids is given and subgroupoids of free groupoids are characterized. For some of these varieties it is shown that the word problem is solvable. The examination of the variety of groupoids defined by the identity $x^n = x$, led to the notion of a groupoid power. For some of these varities the so called Bruck's Theorem is obtained, i.e. the class of free groupoids in the variety is characterized by the injective groupoids in the same variety.

Finally I will say that Academician Čupona treated everyone equally, regardless of nation, race and religion. In his heart, there was love for everyone. He was extremely modest and conscientious man, but also a man who knew how to point out our mistakes. It was a great privilege to be his student and collaborator, for which I am infinitely grateful. Many of my actions today are similar to his.

Academician Čupona died on December 16, 2009, which is actually a great symbolism. The Faculty of Philosophy was established on December 16, 1946, from which later grew FNSM – that was Čupona's second home.

БИБЛИОГРАФИЈА НА АКАДЕМИК ЃОРЃИ ЧУПОНА
BIBLIOGRAPHY OF ACADEMICIAN GJORGJI ČUPONA

[1] Ѓ. Чупона, За еден вид полиња со конечна карактеристика, *Билтен ДМФ НРМ,* Скопје, **6** (1955), pp. 44−46.

[2] Ѓ. Чупона, За еден вид конечни некомутативни групи, *Билтен ДМФ НРМ,* Скопје, **7** (1956), pp. 42−43.

[3] Ѓ. Чупона, За релацијата дистрибутивност меѓу алгебарските операции, *Год. збор. Филоз. фак., Прир.-мат. оддел,* Скопје, **9** (1956), pp. 23−29.

[4] Ѓ. Чупона, За тернарните асоцијативни операции, *Билтен ДМФ НРМ*, Скопје, **9** (1958), pp. 5−10.

[5] Ѓ. Чупона, За редуцибилните полугрупи, *Год. збор. Филоз. фак., Прир.-мат. оддел,* Скопје, **11** (1958), pp. 19−27.

[6] Ѓ. Чупона, За финитарните операции, *Год. збор. ПМФ*, Скопје, **12** (1959), pp. 7−49.

[7] Ѓ. Чупона, За некои релации меѓу бинарните операции, *Билтен ДМФ НРМ,* Скопје, **10** (1959), pp. 5−27.

[8] Ѓ. Чупона, За периодичните полиња, *Билтен ДМФ НРМ,* Скопје, **11** (1960), pp. 5−8.

[9] Ѓ. Чупона, За дистрибутивните системи, *Год. збор. ПМФ,* Скопје, **13** (1960), pp. 5−15.

[10] Б. Трпеновски, Ѓ. Чупона, Финитарни асоцијативни операции со неутрални елементи, *Билтен ДМФ НРМ,* Скопје, **12** (1961), pp. 15−24.

[11] Ѓ. Чупона, За n-арните подполугрупи, *Билтен ДМФ НРМ,* Скопје, **12** (1961), pp. 5−13.

[12] Ѓ. Чупона, За асоцијативните конгруенции, *Билтен ДМФ НРМ,* Скопје, **13** (1962), pp. 5−12.

[13] Ѓ. Чупона, За некои компатибилни фамилии полугрупи, *Год. збор. ПМФ,* Скопје, **14** (1963), pp. 5−14.

[14] Gj. Čupona, Semigroups in which some left ideal is a group, *Annual Bulletin, Faculty of Math. and Nat. Sci.,* Skopje, **14** (1963), pp. 15−17.

[15] Gj. Čupona, On completely simple semigroups, *Glasnik mat. fiz. i ast.,* Zagreb, **18** (1963), pp. 159−164.

[16] Gj. Čupona, *On semigroups S in which each proper subset Sx is a group*, Glasnik mat. fiz. i ast., Zagreb, **18** (1963), pp. 165−168.

[17] Ж. Мадевски, Б. Трпеновски, Ѓ. Чупона, За инфинитарните асоцијативни операции, *Билтен ДМФ СРМ,* Скопје, **15** (1964), pp. 19−22.

[18] Ѓ. Чупона, Полугрупи генерирани од асоцијативи, *Год. збор. ПМФ*, Скопје, **15** (1964), pp. 5−25.

[19] Ѓ. Чупона, За [m,n]-прстените, *Билтен ДМФ СРМ,* Скопје, **16** (1965), pp. 5−10.

[20] Gj. Čupona, On some primitive classes of universal algebras, *Matematički vesnik/Math. Gazzete*, Beograd/Belgrade, **3** (**18**) (1966), pp. 105−108.

[21] Ѓ. Чупона, Подалгебри на полугрупи, *Билтен ДМФ СРМ,* Скопје, **19** (1968), pp. 9−16.

[22] Gj. Čupona, Finitarne asocijativne operacije, *Matematička biblioteka,* Beograd, **39** (1969), pp. 135−149.

[23] Ѓ. Чупона, За асоцијативите, *Прилози, прир.-мат. науки,* МАНУ, Скопје, **I**, 1 (1969), pp. 9−20.

[24] Ѓ. Чупона, Асоцијативи со кратење, *Год. збор. ПМФ,* Скопје, **19** (1969), pp. 5−14.

[25] Ѓ. Чупона, За квазипрстените, *Билтен ДМФ СРМ,* Скопје, **20** (1969), pp. 19−22.

[26] Б. Трпеновски, Ѓ. Чупона, [m,n]-групоиди *Билтен ДМФ СРМ,* Скопје, **21** (1970), pp. 19–29.

[27] Ѓ. Чупона, За теоремата на Кон-Ребане, *Год. збор. ПМФ,* Скопје, **20** (1970), pp. 5−14.

[28] Ѓ. Чупона, За алгебрите на сместување, *Год. збор. ПМФ,* Скопје, **20** (1970), pp. 15−24.

[29] Ѓ. Чупона, Сместување на тополошки алгебри во тополошки полугрупи, *Билтен ДМФ СРМ,* Скопје, **21** (1970), pp. 37−42.

[30] Gj. Čupona, On topological n-groups, *Bulletin of Macedonian Math-Phys. Society,* Скопје, **22** (1971), pp. 5−10.

[31] Ѓ. Чупона, Една класа делумни алгебри, *Год. збор. ПМФ,* Скопје, **22** (1972), pp. 5−37.

[32] Ѓ. Чупона, М-асоцијативи во кои некои леви идеали се М-подгрупи, *Год. збор. ПМФ,* Скопје, **22** (1972), pp. 39−49.

[33] Ѓ. Чупона, Н. Целакоски, On representation of n-associatives into semigroups, *Contributions, Sec. Nat. Math. Sci.,* MASA, **VI**, 1 (1974), pp. 23–34.

[34] Gj. Čupona, S. Markoski, On quasigroups, *Annual. Bulletin, Faculty of Math. and Nat. Sci.,* Skopje, **25-26** (1975-76), pp. 9−14.

[35] Ѓ. Чупона, С. Маркоски, Сместување на универзални алгебри, *Год. збор. ПМФ,* Скопје, **25-26** (1975-76), pp. 15−34.

[36] Gj. Čupona, On a representation of algebras in semigroups, *Contributions, Sec. Nat. Math. Sci.,* MASA, **X**, 1 (1978), pp. 5−18.

[37] Gj. Čupona, On n-groupoids, *Math. Bulletin,* **2** (**28**) (1978), pp. 5−11.

[38] Gj. Čupona, n-subsemigroups of periodic semigroups, *Annual Bulletin, Math, Fac.,* Skopje, **29** (1978), pp. 21−25.

[39] Gj. Čupona, n-subsemigroups of semigroups satisfying the identity $x^r = x^{r+m}$, *Annual Bulletin, Math. Fac.,* Skopje, **30** (1979), pp. 5−14.

[40] Gj. Čupona, G. Vojvodic, S. Crvenkovic, Subalgebras of semilattices, *Review of Research, Faculty of Sci., Univ. of Novi Sad,* **10** (1980), pp. 191−195.

[41] Gj. Čupona, J. Ušan, Z. Stojaković, Multiquasigroups and some related structures, *Contributions, Sec. Math. Tech. Sci.,* MASA, **I**, 2 (1980), pp. 5−12.

[42] Gj. Čupona, N. Celakoski, Polyadic subsemigroups of semigroups, *Proc. Algebraic Conference,* Skopje (1980), pp. 131−152.

[43] Gj. Čupona, G. Vojvodić, S. Crvenković, Subalgebras of commutative semigroups satisfying the law $x^r = x^{r+m}$, *Review of Research, Faculty of Sci., Univ. of Novi Sad,* **11** (1981), pp. 217−230.

[44] Gj. Čupona, Z. Stojaković, J. Ušan, On finite multiquasigroups, *Publications de l'Institut mathéatique, Nouvelle série,* **29** (**43**) (1981), pp. 53−59.

[45] Gj. Čupona, n-subsemigroups of some commutative semigroups, *Annual Bulletin, Math. Fac.,* Skopje, **32** (1981), pp. 33−36.

[46] Gj. Čupona, S. Markovski, Polynomial subalgebras, *Proc. Algebraic Conference, Novi Sad 1981,* (1982), pp. 1−6.

[47] Gj. Čupona, G. Vojvodic, S. Crvenovic, Subalgebras of Abelian torsion groups, *Proc. Algebraic Conference, Novi Sad 1981,* (1982), pp. 141−148.

[48] Gj. Čupona, N. Celakoski, Representations of unary algebras in unars, *Contributions, Sec. Math. Tech. Sci.,* MASA, **III**, 2 (1982), pp. 5−12.

[49] Gj. Čupona, S. Markovski, B. Janeva, The problem of solvability of polylinear representations of universal algebras in semigroups, *Annual Bulletin, Math. Fac.,* Skopje, **33-34** (1982-83), pp. 63−68.

[50] Ѓ. Чупона, Проширувања на алгебарски структури, *Пристапни предавања,* МАНУ, Скопје, (1982), pp. 249−274.

[51] Gj. Čupona, N. Celakoski, A. Samardžiski, Post theorems for unary algebras, *Proc. Symp. n-ary Structures,* Skopje, (1982), pp. 71−84.

[52] Gj. Čupona, N. Celakoski, A. Samardžiski, Roots of permutations, *Proc. Symp. n-ary Structures,* Skopje, (1982), pp. 85−95.

[53] Gj. Čupona, S. Crvenković, G. Vojvodić, Representations of semigroups of operations in semigroups of left translations, *Proc. Symp. n-ary Structures,* Skopje, (1982), pp. 209−222.

[54] Gj. Čupona, A set of semigroup n-varieties, *Proc. Third Alg. Conf. Beograd 1982,* Novi Sad Inst. of Math., (1983), pp. 59−66.

[55] Gj. Čupona, On semigroups of operations, *Contributions, Sec. Math. Tech. Sci.,* MASA, **IV**, 1 (1983), pp. 43−47.

[56] Gj. Čupona, Vector valued semigroups, *Semigroup forum,* **26** (1983), pp. 65−74.

[57] Gj. Čupona, N. Celakoski, Embeddings of associatives in semigroups, *Proc. II Inter. Symp. n-ary Structures,* Varna 1983, Center of Applied Mathematics (1984), pp. 71−78.

[58] Gj. Čupona, S. Markovski, S. Crvenković, G. Vojvodić, Subalgebras of cancellative semigroups, *Proc. II Inter. Symp. n-ary Structures,* Varna 1983, Center of Applied Mathematics (1984), pp. 49−64.

[59] Gj. Čupona, D. Dimovski, On a class of vector valued groups, *Proc. Conf. "Algebra and Logic", Zagreb 1984*, Novi Sad Inst. of Math., (1985), pp. 29−37.

[60] Gj. Čupona, S. Markovski, Cohn-Rebane Theorem for vector valued algebras, *Contributions, Sec. Math. Tech. Sci.*, MASA, **VI**, 2 (1985), pp.143−149.

[61] Gj. Čupona, S. Markovski, Vector valued subgroupoids of semigroups, *Contributions, Sec. Math. Tech. Sci.*, MASA, **VII**, 2 (1986), pp. 5−12.

[62] Gj. Čupona, D. Dimovski, A Samadziski, Fully commutative vector valued groups, *Contributions, Sec. Math. Tech. Sci.*, MASA, **VIII**, 2 (1987), pp. 5−17.

[63] Gj. Čupona, S. Markovski, B. Janeva, Post theorem for vector valued semigroups, *Proc. Conf. "Alg. and Logic" Cetinje 1986*, Novi Sad Inst. of Math., (1987), pp. 33−46.

[64] Gj. Čupona, N. Celakoski, S. Markovski, D. Dimovski, Vector valued groupoids, semigroups and groups, *Vector Valued Semigroups and Groups*, MASA, Skopje, (1988), pp. 1−78.

[65] Gj. Čupona, S. Markovski, D. Dimovski, B. Janeva, Introduction to combinatorial theory of vector valued semigroups, *Vector Valued Semigroups and Groups*, MASA, Skopje, (1988), pp. 141−183.

[66] Gj. Čupona, N. Celakoski, Multidimensional associatives, *Math. Bulletin*, **13** (**39**) (1989), pp. 5−14.

[67] Gj. Čupona, S. Markovski, Polyadic operations induced by semigroups, *Math. Bulletin*, **13** (**39**) (1989), pp. 15−24.

[68] Gj. Čupona, N. Celakoski, A Samardziski, Fully commutative vector valued groupoids, *Proc. Conf. "Alg. and Logic" Sarajevo 1987*, Novi Sad Inst. of Math., (1989), pp. 29−41.

[69] Gj. Čupona, N. Celakoski, Transformations of Booleans, *Math. Bulletin*, **13** (**40**) (1990), pp. 15−26.

[70] Gj. Čupona, S. Markovski, Vector valued groupoids induced by varieties of semigroups, *Math. Bulletin*, **14** (**40**) (1990), 27−38.

[71] Gj. Čupona, B. Janeva, On free objects in some classes of finite subset structures, *Contribution to General Algebra*, **7**, Wien (1991), pp. 75−85.

[72] Gj. Čupona, R. Sz. Madarasz, On poly-algebras, *Review of Research, Faculty of Science, Mathematics Series, Univ. of Novi Sad*, **21** 2 (1991), pp. 141−156.

[73] Gj. Čupona, S. Markovski, Embeddings of universal algebras in semigroups, *Math. Bulletin*, **15** (**41**) (1991), pp. 5−16.

[74] Gj. Čupona, S. Markovski, Free objects in the class of vector valued groupoids induced by semigroups, *Contributions to General Algebra*, **7**, Wien (1991), pp. 87−95.

[75] Gj. Čupona, S. Ilic, Free hypersemigroups, *Contributions, Sec. Math. Tech. Sci.*, MASA, **XIV**, 2 (1993), pp. 13−26.

[76] Gj. Čupona, R. Sz. Madarasz, Free poly-algebras, *Review of Research, Faculty of Science, Mathematics Series, Univ. of Novi Sad*, **23**, 2 (1993), pp. 245−261.

[77] Gj. Čupona, S. Markovski, Free objects in primitive varieties of n-groupoids, *Publ. de l'Inst. Math., Nouvelle série*, **57** (**71**) (1995), pp. 147−154.

[78] Gj. Čupona, S. Markovski, Ž. Popeska, Primitive n-identities, *Contributions to General Algebra* **9**, Wien (1995), pp. 107−116.

[79] Gj. Čupona, N. Celakoski, Free objects in some varieties of groupoids, *Math. Bulletin*, **20** (**46**) (1996), pp. 5−16.

[80] Gj. Čupona, N. Celakoski, Free groupoids with $(xy)^2 = x^2y^2$, *Contributions, Sec. Math. Tech. Sci.*, MASA, **XVII**, 1-2, (1996), pp. 5−17.

[81] Gj. Čupona, S. Markovski, Primitive varieties of algebras, *Algebra Universalis*, **38** (1997), pp. 226−234.

[82] Gj. Čupona, N. Celakoski, Free groupoids with $xy^2 = xy$, *Math. Bulletin*, **21** (**47**) (1997), pp. 5−16.

[83] Gj. Čupona, N. Celakoski, On groupoids with the identity $x^2y^2 = xy$, *Contributions, Sec. Math. Tech. Sci.*, MASA, **XVIII**, 1-2, (1997), pp. 5−15.

[84] Gj. Čupona, N. Celakoski, Free groupoids with $x^n = x$, *Proc. First Congress of Math. and Inf. of Macedonia, 1996*, (1998), pp. 19−25.

[85] Gj. Čupona, S. Ilic, Free groupoids with $x^n = x$ II, *Novi Sad Journal of Math.* **29** 1 (1999), pp. 147−154.

[86] Gj. Čupona, N. Celakoski, B. Janeva, Free groupoids with axioms of the form $x^{m+1}y = xy$ and/or $xy^{m+1} = xy$, *Novi Sad Journal of Math.*, **29** 2 (1999), pp. 131−147.

[87] Gj. Čupona, N. Celakoski, B. Janeva, Some applications of groups in investigations of (n, m)-groups, *Contributions, Sec. Math. Tech. Sci.*, MASA, **XX**, 1−2 (1999), pp. 5−11.

[88] Gj. Čupona, N. Celakoski, B. Janeva, Canonical groupoids with $x^my^n = xy$, *Math. Bulletin*, **23** (**49**) (1999), pp. 11−18.

[89] D. Dimovski, Gj. Čupona, Injective vector valued semigroups, *Novi Sad Journal of Math.* **29** 2 (1999), pp. 149−161.

[90] Gj. Čupona, N. Celakoski, B. Janeva, On fully commutative (n, m)-groupoids, *Contributions, Sec. Math. Tech. Sci.*, MASA, **XXI**, 1−2 (2000), pp. 5−14.

[91] Gj. Čupona, N. Celakoski, S. Ilic, Groupoid powers, *Math. Bulletin*, **25** (**51**) (2001), pp. 5−12.

[92] Gj. Čupona, N. Celakoski, B. Janeva, Varieties of groupoids with axioms of the form $x^{m+1}y = xy$ and

/ or $xy^{n+1} = xy$, *Math. Gazzete /Glasnik Matematički*, Zagreb **37** (**57**) (2002), pp. 235−244.

[93] Gj. Čupona, N. Celakoski, S. Ilic, On monoassociative groupoids, *Math. Bulletin*, **26** (**52**) (2002), pp. 5−16.

[94] S. Ilic, N. Celakoski, Gj. Čupona, Slobodni f-idempotentni grupoidi, *Proc. Second Congress of Math. and Inf. of Macedonia, 2000*, (2003), pp. 31−34.

[95] Gj. Čupona, V. Celakoska-Jordanova, On a variety of groupoids of rank 1, *Proc. Second Congress of Math. and Inf. of Macedonia 2000*, (2003), pp. 17−23.

[96] Gj. Čupona, N. Celakoski, B. Janeva, Injective groupoids in some varieties of groupoids, *Proc. Second Congress of Math. and Inf. of Macedonia 2000*; (2003), pp. 47−55.

УЧЕБНИЦИ
TEXTBOOKS

[1] Ѓ. Чупона, *Предавања по алгебра. Книга I*. – Универзитет Скопје, 1968.

[2] Ѓ. Чупона, Б. Трпеновски, Н. Целакоски, *Предавања по виша математика Книга I*, Универзитет „Кирил и Методиј“, Скопје, 1971, 2-изд. 1976, 3-изд. 1981, 4-изд. 1984.

[3] Ѓ. Чупона, Б. Трпеновски, Н. Целакоски, *Предавања по виша математика Книга II*, Универзитет „Кирил и Методиј“, Скопје, 1971, 2-изд. 1976, 3-изд. 1981, 4-изд. 1984.

[4] Ѓ. Чупона, Б. Трпеновски, Н. Целакоски, *Предавања по виша математика Книга III*, Универзитет „Кирил и Методиј“, Скопје, 1972, 2-изд. 1977, 3-изд. 1981, 4-изд. 1984.

[5] Ѓ. Чупона, *Предавања по алгебра. Книга I*. – Универзитет „Кирил и Методиј“, Скопје, 1972.

[6] Ѓ. Чупона, Б. Трпеновски, *Предавања по алгебра Книга II*, Универзитет „Кирил и Методиј“, Скопје, 1973.

[7] Ѓ. Чупона, *Алгебарски структури и реални броеви,* Просветно дело, Скопје, 1976.

[8] Б. Трпеновски, Н. Целакоски, Ѓ. Чупона, *Виша математика Книга II*, Просветно дело, 1993.

[9] Б. Трпеновски, Н. Целакоски, Ѓ. Чупона, *Виша математика Книга III*, Просветно дело, 1994.

[10] Б. Трпеновски, Н. Целакоски, Ѓ. Чупона, *Виша математика Книга VI*, Просветно дело, 1994.

[11] Б. Трпеновски, Н. Целакоски, Ѓ. Чупона, *Виша математика Книга I*, Просветно дело, 1995.

[12] Ѓ. Чупона, *Предавања по алгебра. Книга I*. – Универзитет „Св. Кирил и Методиј“, Скопје, 2000.

[13] Ѓ. Чупона, Б. Трпеновски, *Предавања по алгебра Книга II*, Универзитет „Св. Кирил и Методиј“, Скопје, 2000.

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CONTRIBUTIONS, Section of Natural, Mathematical and Biotechnical Sciences, MASA, Vol. **41,** No. 2, pp. 87–90 (2020)

ОМАЖ ЗА ЃОРЃИ ЧУПОНА

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Еден и еден и еден се три
Дојдете заедно, „Битлси", 1969

Македонската академија на науките и уметностите, во 2020 година, ја одбележува деведесет годишнината од раѓањето на академик Ѓорѓи Чупона. Тој беше човек со повеќе димензии: интелектуалец, математичар, филозоф, визионер, ерудит, планинар. Луѓето се прашуваа, кога ќе го видеа во кола, како беше возможно колата да ја собере „таа математичка и духовна величина" [1].

Роден во Маловиште, во близина на Битола, академик Чупона го поминува детството во Јени Маале, (на македонски Нова Улица), една од најстарите населби во Битола, на источната страна на градот. Како дете, во Јени Маале, Ѓорѓи сакал да игра ашици (игра со мали коски од овци), а токму тоа бил адутот на Трајан Чамо: „Таа своја увереност Чамо ја засноваше на тоа што Чупона не само што беше битолчанец туку и јенимаалец, а партијата, главно, го регрутираше своето членство од Јени Маале. Но најголемиот адут во неговите раце бил тоа што заедно со него како деца играле ашици" [1].

Гимназија завршил во Битола, во „Гоце Делчев" – првата гимназија во ослободена Македонија во која на 6 февруари 1945 година почнува да се изведува настава на македонски јазик. Во 1952 година, со одлука на Наставничкиот совет, името е променето во „Јосип Броз Тито", а тоа име го носи и денес. Професор Чупона „уште како ученик во гимназијата, при сменувањето на нејзиното име од 'Гоце Делчев' во 'Јосип Броз Тито', јавно изјави дека тоа што е Тито за Македонија, во свое време бил и Гоце Делчев. На исто рамниште ги стави" [1]. Поради своите јавни настапи, Ѓорѓи беше повикуван на одговорност. „Да е само тоа! – имаше и друга забелешка Борис Чакре за сметка на предлогот на неговиот опонент. – При едно скандирање 'Тито-партија', тој, место Тито, извикуваше 'Мито'. Јавно го зедоа на одговорност. 'Ако е Тито народен човек, без некое педигре, не е навреда за него да се спореди со Мито, тој народен човек, еснаф', беа зборовите на Чупона" [1].

Академик Чупона зборуваше за разликата меѓу параболите во религијата (кратки метафорични и поучни раскази со кои Господ Исус Христос се обраќа кон народот и апостолите) и параболите во математиката (вид криви линии). „Вие, господине Ѓорѓи, ќе дозволите да ви забележам, скршнавте од 'линијата' на партијата. Вие бевте 'парабола', така и ве нарекуваме. 'Тоа сум и сега!' – категорично изјави Чупона. – 'Ама како граѓанин, а не како математичар. Во математиката нема лево-десно, Во математиката два и два се четири'" [1].

Академик Чупона беше математичар. Математиката има чудотворна моќ: „Тогаш, мој благороден пријателу, геометријата ќе ја повлече душата кон вистината, и ќе создаде дух на филозофијата и ќе го подигне она што сега е несреќно дозволено да падне" [2].

Во 1623 година, Галилео Галилеј забележа дека универзумот е голема книга, напишана на математички јазик. Оние сиромашни души кои не го разбираат тој јазик, предупреди тој, талкаат низ „темниот лавиринт" [3].

Чупона докторираше многу млад, на 29 години, со темата „Прилог кон теоријата на алгебарските структури", го сметаат за основоположник на алгебрата во Македонија, како и на дисциплините тесно поврзани со неа. Тој е првиот македонски математичар кој објавил научни трудови од областа алгебра.

Зборот алгебра има корен во арапскиот збор Al-Jabr. Во IX век, персискиот математичар al-Khwarizmi (780 – 850) напиша книга, која претставуваше пресвртница во историјата на математиката, воспоставувајќи ја дисциплината алгебра [4]. Изразот „алгебра" произлезе од насловот на книгата. Во насловот, al-jebr w' almuqabala, зборот al-jebr означува префрлање на величина од едната страна на равенката на друга, додека muqabala значи поедноставување на добиените изрази. Фигуративно, al-jebr значи враќање на рамнотежата во равенката [5].

Методите за решавање линеарни и квадратни равенки беа познати уште од времето на Вавилон, но за комплетното решение на равенките од трет степен (кубни равенки) се чекаше до XVI век. Во 1202 година, италијанскиот математичар Фибоначи (1170 – 1240) објави книга, во која, во Европа, го воведе хинду-арапскиот броен систем. Решенијата на равенките од трет и четврт степен првпат беа објавени во книгата Ars Magna (во 1545 година) од Кардано (1501 – 1576), иако тој не откри ниту една од двете соодветни формули [4]. Формулата за равенките од трет степен ја пронајде Фонтана (1500 – 1557), попознат со прекарот Tartaglia (или Пелтачец – од пелтечи), додека равенките од четврти степен ги реши ученикот на Кардано, Ферари (1522 – 1565).

Кардано – лекар, математичар, коцкар и ексцентрична личност пар екселанс – го објави решението на Tartaglia во својата книга и покрај тоа што вети, под заклетва, дека тоа нема никогаш да го направи. Во тоа време, учеството на математички натпревари беше начин на живеење. На овие натпревари се победуваше врз основа на брзината на решавање кубни равенки, па затоа методите за изнаоѓање брзи решенија се чуваа во строга тајност. Чинот на предавство резултираше во доживотна расправија меѓу двајцата математичари [4].

За изнаоѓање на решенијата на равенки од степен поголем од четири се чекаше до XIX век. Сепак, методите, од вториот век пред нашата ера, за решавање на системите на линеарни равенки со употреба на матрици и детерминанти, повторно се појавија приближно кон крајот на XVII век и во тоа време започна развојот на областа што денес се нарекува линеарна алгебра.

Во раниот XIX век, францускиот математичар Галоа (1811 – 1832) докажа дека не постои општа формула за решавање равенки од степен поголем од четири. Неговите резултати придонесоа за развој на нова област на математиката, теоријата на групи. Галоа почина на 20-годишна возраст во дуел. Се верува дека, ноќта пред фаталниот настан, се напишани многу од неговите математички откритија, идеи што биле целосно разбрани дури 100 години по неговата смрт. На маргините на неговите белешки се запишани зборовите: „Немам време!“ [4].

За зборот алгебра, професорот Morris Kline, познат по тоа што пишуваше за историјата и филозофијата на математиката, ќе напише: „Кога Маврите стигнаа до Шпанија... алгебрист (на шпански algebrista)... значеше човек што мести дислоцирани коски... и над шпанските берберници беа пронајдени знаци на кои пишуваше Algebrista y Sangrador (што во слободен превод значи човек што мести коски и пушта крв). Така, може да се каже дека постои добра историска основа за фактот дека зборот алгебра предизвикува немили мисли“ [5]. Michael Atiyah, претседател на Кралското друштво во период од 1990 до 1995 година, математичар – специјалист во геометрија, ќе напише: „Алгебра е понудата на ѓаволот до математичарот. Ѓаволот вели: Јас ќе ти ја дадам оваа моќна машина, таа ќе одговори на секое прашање што го сакаш. Сè што треба да направиш е да ми ја дадеш душата: откажи се од геометријата и ќе ја имаш оваа прекрасна машина“ [6].

Професор Чупона беше визионер. Голем е неговиот ангажман и во развојот на компјутерските науки во Македонија, посебно на делот тесно поврзан со математиката. Тој ги иницираше првите истражувања во вештачка интелигенција. За напредокот на вештачката интелигенција можеби најдобро ќе посведочат зборовите на Steven Strogatz, професор по применета математика на Универзитетот Корнел во САД. Споредувајќи ја шаховската игра на машината AlphaZero во 2018 година со играта на машината Deep Blue, која во далечната 1997 година го победи тогашниот актуелен шампион Гари Каспаров, Strogatz пишува [7]: „И во подобро и во полошо, Deep Blue играше како машина, брутално и материјалистички. Можеше да го надмине господин Каспаров во пресметување, но не и во мислење“. Сепак, тој понатаму потенцираше: „Најразочарувачки (за другите играчи, опонентите, моја забелешка) беше тоа што AlphaZero изгледаше како да покажува разбирање. Играше како ниеден друг компјутер дотогаш, интуитивно и убаво, со романтичен, напаѓачки стил“.

За Анри Поанкаре (Henri Poincaré), математиката е „уметност да им се дава исто име на различни нешта“. На сличен начин, поетите создаваат слоеви на значење користејќи зборови и слики што имаат повеќе толкувања и асоцијации. И математичарите и поетите се стремат кон економичност и прецизност во изразувањето, избирајќи ги точно потребните зборови за да го пренесат нивното значење.

Ќе го завршам овој омаж за академик Чупона со делови од песната „Ода на броевите“ од Пабло Неруда [8]:

О, жедта да се знае / колку! / Гладот / да се знае / колку / ѕвезди на небото!

Ние го потрошивме / детството броејќи / камења и растенија, прсти на рацете и / нозете, зрна песок и заби, / нашата младост помина пребројувајќи / ливчиња и опашки на комети.

ЛИТЕРАТУРА – REFERENCES

[1] В. Костов. *„Омаж за математичарот Ѓорѓи Чупона“*, расказ од Владимир Костов, *Даровна*. Скопје: Матица македонска, 2012.

[2] B. Jowett, *Plato: The Republic,* 1871, http://classics.mit.edu/Plato/republic.8.vii.html.

[3] S. Drake, *Discoveries and Opinions of Galileo* (Doubleday & Co., New York), 1957.

[4] S. Glaz, Poetry inspired by mathematics: a brief journey through history, *Journal of Mathematics and the Arts*, **5:4**, (2011), pp. 171–183

[5] M. Kline, *Mathematics and the Physical World*, Dover Publications, 1959.

[6] M.Atiyah, *Collected works*, Vol. **6**, Oxford Science Publications, The Clarendon Press Oxford University Press, 2004.

[7] S. Strogatz, *One Giant Step for a Chess-Playing Machine*, The New York Times, Dec. 26, 2018, https://www.nytimes.com/2018/12/26/scence/chess artificial-intelligence.html.

[8] P. Neruda, *„Ode to Numbers”*, in Elemental Odes, Libris, 1991.

HOMAGE FOR GJORGJI ČUPONA

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One and one and one is three.
Come together, The Beatles, 1969

The Macedonian Academy of Sciences and Arts, in 2020, marks the ninetieth anniversary of the birth of Academician Gjorgji Čupona. He was a man of many dimensions: intellectual, mathematician, philosopher, visionary, erudite, mountaineer. People wondered, when they saw him in a car, how it was possible for the car to fit "that mathematical and spiritual greatness” [1].

Born in Malovishte, near Bitola, Academician Čupona spent his childhood in Yeni Maale (English New Street), one of the oldest settlements in Bitola, on the east side of the city. As a child in Yeni Maale, Gjorgji liked to play ‘ašici’ (a game with small sheeps bones), and that was exactly Trajan Čamo’s trump card: “Čamo based his conviction on the fact that Čupona was not only from Bitola, but also from Yeni Maale, and The Party mainly recruited its membership from Yeni Male. But the biggest trump card in his hands was that they played ašici together as children.” [1]

He finished high school in Bitola, in “Goce Delčev” – the first gimnazium in Liberated Macedonia where on February 6, 1945 teaching in Macedonian language began. In 1952, with a decision of the Teachers' Council, the name was changed to “Josip Broz Tito”, and it bears that name today. Professor Čupona “as a student in the gimnazium, during the change of her name from ‘Goce Delčev’ to ‘Josip Broz Tito’, publicly stated that what is Tito for Macedonia, in his time was Goce Delčev. He put them on the same level” [1]. Gjorgji was held accountable for his public appearances. “If only that! – There was another remark by Boris Čakre about the proposal of his opponent – During a chant. ‘Tito-Party’, he shouted ‘Mito’ instead of Tito. He was publicly held accountable. ‘If Tito is a people man, without any pedigree, it is not an insult for him to be compared to Mito, that people’s man, a guild’, were the words of Čupona” [1].

Academician Čupona spoke about the difference between parabolas in religion (short metaphorical and instructive stories with which the Lord Jesus Christ addresses the people and the apostles) and parabolas in mathematics (kind of curves). “You, Mr. Gjorgji, will let me note that you have deviated from the ‘line’ of the Party. You were a “parabola”, that's how we called you. That I am now! - said Čupona categorically. - But as a citizen, not as a mathematician. In mathematics there is no left-right, in mathematics two plus two are four” [1].

Academician Čupona was a mathematician. Mathematics has miraculous powers: “Then, my noble friend, geometry will draw the soul towards truth, and create the spirit of philosophy, and raise up that which is now unhappily allowed to fall down.” [2]

In 1623, Galileo Galilei noted that the universe was a large book, written in mathematical language. Those poor souls who do not understand that language, he warned, wander through the “dark labyrinth”. [3]

Čupona received his Phd degree very young, at the age of 29, with the topic “Contribution to the theory of algebraic structures”. He is considered the founder of algebra in Macedonia, as well as the disciplines closely related to it, he is the first Macedonian mathematician who published scientific papers in the field of algebra.

The word algebra has its roots in the Arabic word Al-Jabr. In the 9th century, the Persian mathematician al-Khwarizmi (780–850) wrote a book that marked a turning point in the history of mathematics, establishing the discipline of algebra [4]. The term "algebra" came out from the title of the book. In the title, al-jebr w 'almuqabala, the word al-jebr means shifting a quantity from one side of the equation to the other, while muqabala means simplifying the resulting expressions. Figuratively, al-jebr means the return of equilibrium in the equation [5].

The methods for solving linear and quadratic equations have been known since the time of Babylon, but the complete solution of the third degree equations (cubic equations) was not found, until the 16th century. In 1202, the Italian mathematician Fibonacci (1170–1240) published a book in which he introduced the Hindu-Arabic number system in Europe. Solutions to the third and fourth degree equations were first published in the book Ars Magna (1545) written by Cardano (1501–1576), although he did not discover either of the two corresponding formulas [4]. The formula for third-degree equations was found by Fontana (1500–1557), better known by the nickname Tartaglia (from the word stutter), while fourth-degree equations were solved by Cardano's student Ferrari (1522–1565).

Cardano - a medical doctor, mathematician, gambler, and eccentric par excellence - published Tartaglia's solution in his book despite promising, under oath, that he would never do it. At that time, participating in math competitions was a way of life. The winning on those competitions was based on the speed of solving cubic equations, so the methods for finding quick solutions were kept strictly secret. The act of betrayal resulted in a lifelong quarrel between the two mathematicians [4].

Solutions to equations of degree greater than four were not found, until the 19th century. However, methods, from the second century BC, for solving systems of linear equations using matrices and determinants, reappeared around the end of the 17th century and at that time the development of the field now called linear algebra started.

In the early 19th century, the French mathematician Galois (1811–1832) proved that there was no general formula for solving equations of degree greater than four. His results contributed to the development of a new field of mathematics, group theory. Galois died at the age of 20 in a duel. It is believed that the night before the fatal event, many of his mathematical discoveries were written, ideas that were not fully understood until 100 years after his death. On the margins of his notes, the words: "I have no time!" are written [4].

About the word algebra, Professor Morris Kline, known for writing about the history and philosophy of mathematics, wrote: "When the Moors arrived in Spain ... an algebraist (in Spanish algebrista) ... meant a man who places dislocated bones ... Algebrista y Sangrador (which in free translation means a man who places bones and bleeds), were words written on signs found above Spanish barbershops. Thus, it can be said that there is a good historical basis for the fact that the word algebra evokes unpleasant thoughts" [5]. Michael Atiyah, president of the Royal Society from 1990 to 1995, a mathematician specializing in geometry, wrote: "Algebra is the devil's offer to the mathematician. The devil says: I will give you this powerful machine, it will answer any question you want. All you have to do is give me your soul: give up geometry and you will have this wonderful machine" [6].

Professor Čupona was a visionary. His engagement in the development of computer science in Macedonia is great, especially in the part closely related to mathematics. He initiated the first research in artificial intelligence. The advancement of artificial intelligence is perhaps best illustrated by the words of Steven Strogatz, a professor of applied mathematics at Cornell University in the USA. Comparing the chess game of the AlphaZero machine in 2018 with the game of the machine Deep Blue, which in the distant 1997 defeated, then the current champion Gary Kasparov, Strogatz writes [7]: "For better or worse, Deep Blue played as a machine, brutally and materialistically. It could have surpassed Mr. Kasparov in calculation, but not in thinking". However, he added: "The most disappointing (for the other players, the opponents, my remark) was that AlphaZero seemed to show understanding. It played like no other computer until then, intuitively and beautifully, with a romantic, aggressive style."

For Henri Poincaré, mathematics is "the art of giving the same name to different things". Similarly, poets create layers of meaning using words and images that have multiple interpretations and associations. Both mathematicians and poets strive for precision and economy in expression, choosing the exact necessary words to convey their meaning.

I will conclude this homage to Academician Čupona with excerpts from Pablo Neruda's "Ode to numbers" [8]:

Oh. The thirst to know / how many! The hunger / to know / how many/ stars in the sky!

We spent / our childhood counting / stones and plants, fingers and /toes, grains of sand, and teeth, / our youth we passed counting / petals and comets' tail.

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CONTRIBUTIONS, Section of Natural, Mathematical and Biotechnical Sciences, MASA, Vol. **41,** No. 2, pp. 91–96 (2020)

ЃОРЃИ ЧУПОНА – МАТЕМАТИЧАР, УЧИТЕЛ, ВИЗИОНЕР

Димитра Карчицка, Смилка Здравковска

СЕЌАВАЊА ЗА ПРОФЕСОР ЃОРЃИ ЧУПОНА – УЧИТЕЛ СО ГОЛЕМ УГЛЕД НА ГЕНЕРАЦИИ МАТЕМАТИЧАРИ И ИНФОРМАТИЧАРИ

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Математиката, општо, и алгебрата, посебно, претставуваа животна преокупација на професорот Ѓорѓи Чупона, којашто довела до развој на пошироко познатата Македонска алгебарска школа во 70-тите, 80-тите и 90-тите години на минатиот век. Во тој период бил создаден соодветен кадар, биле организирани специјалистички и магистерски студии, биле издадени универзитетски учебници, започна печатењето на стручни и научни списанија, и беа воведени современи математички дисциплини во наставните планови на Природно-математичкиот факултет (ПМФ), потоа на Математичкиот факултет и на обновениот ПМФ во 1985 година.

Професорот Чупона имаше визија, идеи, даваше иницијативи и наоѓаше начини за нивно остварување. Умееше да насети, да препознае, да открие талентирани ученици, студенти и за нив да овозможи организирање математички школи и издавање соодветни помагала за математички натпревари.

Посебно внимание и помош за проширување на знаењата, за усовршување и соодветно насочување во науката им придаваше на успешно дипломираните студенти со организирање математички кружоци, научни собири и конференции во Македонија со учество на еминентни странски научници. Ги следеше постигнатите резултати, ги бодреше соработниците во нивното напредување. Притоа, со сите секогаш беше отворен, искрен, директен, останувајќи МАТЕМАТИЧАР за пример и УЧИТЕЛ во најубавата смисла на зборот. Како необично скромен човек, професорот Чупона избегнуваше формалности, конвенционалности, не сакаше почести, признанија, искажување благодарност кон него, иако најмногу ги заслужуваше.

Имав среќа и привилегија на почеток да бидам студент на професорот Чупона, а потоа речиси сите години на мојата активност како математичар да бидам една од неговите соработници на Факултетот и на Институтот, да го минам патот од асистент до редовен професор по линеарна алгебра и математичко програмирање со негова поддршка и помош. Професорот Чупона имаше увид во мојата наставна, стручна и научна работа. По разгледувањето на моите пишани трудови, ми даваше предлози за исправки, подобрувања, дополнувања, без да инсистира на нивно прифаќање. Таков беше случајот и со мојот последен ракопис „Конечно димензионални векторски простори во задачи со предложени решенија“. Во улога на рецензент даде многубројни, детално обмислени корисни предлози и, на крајот, ми напиша: „Во случајот се работи за фактот што речиси секогаш, дури и несвесно, се обидувам да најдам недостатоци“. Не сум сретнала друг човек толку искрен и спремен да ги преиспитува своите ставови и да прифаќа поинаков пристап и гледиште.

Големата поддршка и ненаметливата помош од моите професори, посебно од професорот Чупона, придонесоа да стекнам самодоверба во работата, да се соочувам со тешкотии, проблеми, да прифаќам обврски, задолженија, функции на Факултутетот, на Институтот, и во рамките на можностите да ги извршувам совесно.

При формирањето на Институтот за информатика (ИИ), покрај Институтот за математика на обновениот ПМФ во 1985 година, дојде до израз големиот авторитет на професорот Чупона и неговата доминатна улога во создавањето наставно-научен кадар од областа на современите науки и во брзиот и успешен развој во дејноста на ИИ. Наведените конкретни податоци подолу укажуваат на ова. Во периодот 1985/86 – 1994/95 година, на студиите по информатика на ПМФ дипломираа 162 студенти. Во почетокот, кадарот на ИИ беше составен од двајца редовни професори, двајца доценти, тројца виши предавачи, двајца асистенти и четворица помлади асистенти, повеќето од нив математичари. По 10 години, кадарот на ИИ, организиран во Завод за теориски основи на информатиката и Завод за компјутерски науки и информатика, го сочинуваа тројца редовни професори (математичари), петмина вонредни професори (тројца доктори на математички науки и двајца доктори на информатички науки), четворица доценти (сите доктори од областа на информатичките науки), четворица асистенти (двајца магистри математичари и двајца магистри информатичари), тројца помлади асистенти и шестмина стручни соработници, а во Сметачкиот центар на ИИ работеа систем-инженер – програмер, оператор и технички секретар дактилограф. Првите доктори од областа на информатиката на ИИ набргу заминаа на специјализација на странски универзитети и, за жал, не се вратија назад.

Лично, како математичар, чувствувам голем долг кон професорот Чупона, кој беше во почетокот на својата академска кариера за време на моите студии (1957/58 – 1960/61) на ПМФ во Скопје, по моето доаѓање од Полска, кога преку математиката учев од моите професори да зборувам, да читам, да пишувам и да размислувам на македонски јазик. Во академската 1960/61 година, прв пат се држеа предавања по новововедениот предмет Алгебра, со содржини од алгебарски структури и логика, покрај дотогашните предмети за студентите од четврта година. Наставата и вежбите му беа доверени на д-р Ѓорѓи Чупона, асистент, за кого претстоел избор во наставно звање. Во јунската испитна сесија се пријавив за полагање дипломски испит, составен од сите математички предмети на последната студиска година, каде што бев единствена од мојата генерација. На денот на испитот, д-р Чупона не бил повикан за присуство, како нов член на испитната комисија, и се случи да добијам оценка за положен дипломски испит без да ми бидат поставени прашања од д-р Чупона за алгебра. Тоа било една од причините зошто д-р Чупона се пријавил на конкурс и бил избран за доцент на Техничкиот факултет во Скопје. Но, и покрај тоа, тој продолжи да држи Алгебра на ПМФ, каде што набргу се врати како вонреден професор за подоцна да биде унапреден во редовен професор на Математичкиот институт на ПМФ.

Со текот на годините, мојот долг кон професорот Чупона сè повеќе растеше. Тој континуирано ја следеше мојата работа како наставник, моите резултати во науката, и како рецензент ги оценуваше и ги вреднуваше нив објективно.

Незаборавни за мене остануваат поземјотресните години во Скопје, но не само по многу сложените услови за живот и работа на Универзитетот. Во тоа време, со колегите асистенти, Александар Самарџиски од ПМФ и Наум Целакоски од Електро-машински факултет, бевме примени на магистерски студии по математика на Белградскиот универзитет како вонредни студенти. Требаше да полагаме испити по четири општи предмети (Увод у алгебарске структуре, Топологија, Анализа 3, Линерна алгебра) пред изборот и полагањето испит по главен предмет и два помошни предмети, различни за секого од нас. Потоа, следуваше избор на тема, изработка и одбрана на магистерскиот труд. Тешкотии се јавуваа бидејќи немавме материјали и соодветни упатства, помагала за нивно подготвување. Професорот Чупона, свесен за потребата од нови кадри и макотрпниот пат за нивно создавање, организираше математички кружок за нас, постдипломците. Еднаш неделно, со негово присуство како раководител, имавме состаноци на кои излагавме научени содржини, дискутиравме, поставувавме прашања и баравме одговори, и наоѓавме задачи за решавање. Професорот активно учествуваше во сè, како и тој самиот да беше пред полагање испити. Кога не се снаоѓавме најдобро, и гледајќи нè како се мачиме со превод од англиски на цела книга, во шега ни рече да бидеме задоволни што се наоѓаме на патот за најобразовани математичари во нашата Република.

Од професорот Чупона научивме да се преиспитуваме пред да даваме одговор на прашање и пред да нудиме решение на задача, а секоја тема да ја анализираме од сите аспекти при нејзино разгледување.

Благодарна сум што ја имам честа при одбележувањето на 90 години од раѓањето на академик Ѓорѓи Чупона, да искажам почит кон неговата личност и неговото дело – фундаментално за македонските науки.

Со верба за подобро утре на сите добри луѓе во светот во ова време-невреме.

СЕЌАВАЊА НА ЃОРЃИ ЧУПОНА

Смилка Здравковска, Ен Арбор, Мичиген, САД

Првите зборови кои ми паѓаат напамет кога сакам да го опишам Чупона (така најчесто му се обраќав по неколку децении познанство) се: човек кој е секогаш подготвен секому да му помогне, а е многу скромен, чесен, директен, визионер и идеалист.

Го запознав кога станав гимназијалка во 1960 година, преку неговата сопруга Горица Илиева, мојата омилена професорка по математика. Таа важеше за строга, но, всушност, само поради тоа што ги поттикнуваше своите ученици повеќе да работат и да научат и секогаш беше подготвена да помогне. Во таа помош го ангажираше и својот сопруг, кој на тој начин откриваше потенцијални идни студенти по математика и други области.

Понекогаш, иако ретко, им одевме дома да ни даваат задачи; и други ми имаат кажано неодамна како Чупона им помагал по математика кај нив дома. Ова ме потсеќа на следново: во 2002 година случајно запознав една жена, на над 90-годишна возраст, многу интересна и културна, Ана Јона, и поминав неколку месеци со неа. Кога разбра дека сум математичар, ми раскажа како кога била ученичка, родителите го замолиле нејзиниот чичко да ѝ помага по математика. И така, младиот но веќе познат математичар Фубини, како што Ана раскажуваше, при часовите одвреме-навреме ќе излезел на нивниот балкон во Торино да си ја куби косата и да вика: "E una bestia!" Тоа беше незамисливо да му се случи на Чупона. Тој имаше, според моето искуство, бескрајно трпение и скромност.

Трудот се исплати; во 1964 година, Југословенската деветчлена екипа на Меѓународната математичка олимпијада во Москва имаше двајца ученици на брачната двојка Илиева-Чупона: Виктор Урумов и јас. Овде да спомнам дека во таа екипа тројца беа од Словенија, четворица од Србија и двајца од Македонија, и дека сме сè уште во повремен контакт; е-пораките ни почнуваат со „Драги олимпијци“. Ми се чини дека би му било мило на Чупона да знае дека екипата која ја претставуваше Југославија сè уште постои како таква.

Виктор и јас отидовме на студии во Москва, по физика и математика, соодветно, благодарение на стипендии од Советскиот Сојуз како помош на Скопје по земјотресот во 1963 година.

Но, визионерската улога на Чупона продолжуваше. По негова иницијатива беше формиран Математичкиот институт со Нумерички центар (МИНЦ) на Универзитетот во Скопје, каде што беше и директор. Идејата му беше да создаде интердисциплинарен центар на Универзитетот, каде што стручњаци од разни области – математика, информатика, економисти, инженери, биолози и други ќе работат заедно. Ова беше многу напредна визија во тоа време.

Сите во МИНЦ беа вработени од Чупона во првите години. Први, примени еден по друг во текот на еден-два месеца во 1969 година, бевме Симеон Иванов, Јосиф Хаџи-Пецов и јас. За простории најде барака во Карпош до Технолошкиот и Машинскиот факултет, па продолжи со вработување на многу други од секакви дисциплини. (Толку многу млади имаше во 1974 година што четири жени се породивме речиси истовремено.) Се надевам дека некој ќе се зафати со пишување историја на тој влијателен институт, поткрепена со документација; јас таква документирана статија барав, но не најдов, иако постојат разни, малку контрадикторни податоци што се објавени.

Чупона беше прв што предаваше топологија во Македонија, предмет за кој имаше напишано многу убава скрипта. Покрај тоа, тој соработуваше со тополози низ целата Југославија, како што се, на пример С. Мардешиќ, Ј. Врабец и други. Но, бидејќи се очекуваше вработените во МИНЦ да предаваат на Универзитетот, тој ми го даде да го предавам курсот по топологија и великодушно дозволи да ги користам неговите белешки. Чупона остана многу активен член на семинарот по топологија и заедно со мене се грижеше за талентираните студенти кои беа заинтересирани за тој предмет. Отпосле разбирам до колкава мера несебично им помагаше на моите студенти, особено по моето заминување во САД. Повеќе за тоа најдобро самите веќе имаат напишано или, пак, би можеле да го сторат тоа во иднина.

Не можам да пишувам за Чупона без да ги спомнам долгите прошетки по планините, особено Водно, и Матка, на кои одеше со своите помлади колеги, кога муабетот не беше само за математика, туку и за политика, општество, природа, живот. Тој беше многу директен, без да се грижи да му се спротивстави на собеседникот, а сепак од него бликаше идеализам, што многу ми се бендисуваше.

Се надевам дека во мала мера, иако многу накратко, ја илустрирав првата реченица од овие сеќавања.

GJORGJI ČUPONA –MATHEMATICIAN, TEACHER, VISIONARY

Dimitra Karčicka, Smilka Zdravkovska

PROFESSOR GJORGJI ČUPONA – A TEACHER WITH A GREAT REPUTATION AMONGST GENERATIONS OF MATHEMATICIANS AND COMPUTER SCIENTISTS

Dimitra Karčicka, PhD, Retired professor, Skopje, Republic of Macedonia

Mathematics and algebra in general have been a lifelong preoccupation of Professor Gjorgi Čupona, which brought to the development of the widely known Macedonian algebra school in the 70s, 80s and the 90s of the last century. During that time appropriate staff was acquired, and specialist and master (graduate) studies were organized. Additionally, university textbooks were published, and the process of printing and publishing of scientific and expert journals was well underway. The Faculty of Natural Sciences and Mathematics (FNSM) also adopted contemporary mathematical disciplines, incorporating them into the already existing syllabi of the Faculty, which is a process that was adopted by the Mathematical Faculty and the renewed Faculty of Natural Sciences and Mathematics, in 1985, as well.

Čupona was a professor with a vision. He was full of ideas and initiatives, always finding opportunities and creating possibilities to realize them. He had a special sense for recognizing talented pupils and students for which he continuously succeeded to organize mathematical lectures and acquire all the corresponding requirements in order to make them present on mathematical competitions.

He was paying a special attention to the newly graduated students as well, providing them his support and help to wider their knowledge and focus more deeply in science. He was continuously organizing mathematical rounds, scientific gatherings and conferences in Macedonia, where he was inviting eminent foreign scientists. Following the new research results and the advances of of his collaborators, he was constantly encouraging them in their work. In addition to all that, he was an open minded, honest and direct person, remaining to be an example of a MATHEMATICIAN in his essence and a TEACHER in the most meaningful way. As an unusually modest person, Professor Čupona used to avoid formalities and conventionalities. He didn't like to receive honors, recognitions nor gratitude expressions, though he had deserved them most of all.

I was lucky and privileged to have been Professor Čupona's student, and later, through all the years of my activity as mathematician, to become his colleague at the Faculty and the Institute, and to go along my academic carrier from a teaching assistant to a full professor of linear algebra and mathematical programming, with his support and help. Professor Čupona had an insight into my teachings and also in my professional and scientific work. When reviewing my scientific papers, he used to give me valuable suggestions for corrections and also ideas for improvements, but still, without insisting on their acceptance. Same thing happened with my last manuscript "Finite-Dimensional Vector Spaces through Problems with Proposed Solutions". As a reviewer, he gave me numerous but carefully thought-out useful suggestions and at the end he wrote: "In this case, it is about the fact that I almost always, even unconsciously, try to find flaws." I have not met such sincere person so far, a person willing to reconsider his opinions and accept a different approach and point of view.

The enormous support and unobtrusive help I have got from my professors, especially from Professor Čupona, made a great impact in my personal development: it helped me to gain a self-confidence in the work, to face various difficulties and solve problems more successfully, but also to accept responsibilities within the Faculty and the Institute, and to perform and act conscientiously, the best I could within my possibilities.

The great authority of Professor Čupona and his leading role in creating teaching and scientific staff in the field of modern sciences, played a crucial role in the establishment of the Institute of Informatics, in addition to the Institute of Mathematics, both within FNSM (renewed in 1985). Moreover, his activities had resulted in rapid and successful development of the Institute of Informatics. The following specific data indicate the above. A total number of 162 students have graduated and successfully finished the studies of Informatics at FNSM in the period 1985/86 - 1994/95. Initially, the staff of the Institute of Informatics consisted of two professors, two assistant professors, three senior lectures, two teaching assistants and four junior assistants, most of them mathematicians. Ten years later, the Institute of Informatics was reorganized in 2 departments:

Department for Theoretical Foundations of Informatics and Department for Computer Science and Informatics. The staff consisted of 3 professors (mathematicians), five associate professors (three of them - PhD holders in mathematics and two of them - PhD holders in computer science), four assistant professors (all of them PhD holders in the field of computer sciences), four teaching assistants (two mathematicians with MSc degree and two IT specialists with MSc degree), three junior assistants and 6 professional associates. There have also been employed a system engineer-programmer, an operator, and a technical secretary-typist at the Institute of Informatics' Computer Center. The first candidates that have obtained a PhD degree in the field of computer sciences left abroad for further specialization at foreign universities, and unfortunately, never came back home.

Personally, as a mathematician, I feel a great debt towards Professor Čupona, whose academic carrier was at the begining during my studies at FNSM in Skopje (1957/58 – 1960/61), after my arrival from Poland, when I learned from my professors and through mathematics, to speak, read, write and think in Macedonian language. Lectures on the newly introduced course Algebra, with contents on algebraic structures and logic, were held for the first time, in the academic year 1960/61. This course was an addition to the already existing courses in the 4th (last) year of studies. The lectures and exercises were assigned to Dr. Gjorgi Čupona, a teaching assistant, who was going to be awarded by an academic teaching title. In the June exam session, I applied for a diploma exam that consisted of all the math courses I have followed in the last academic year of my studies, where I was the only one of my generation. Dr. Čupona was not invited to attend my diploma exam as a new member of the examination committee, and thus it happened that I have received a grade for passing the diploma exam, without answering questions on algebra contents, which were supposed to be given by Dr. Čupona. That was one of the reasons that made Dr. Cupona to apply for a position of an assistant professor at the Technical Faculty in Skopje, and consequently to get that position. Nevertheless, he continued to give Algebra classes at FNSM, where he soon returned as an associate professor, and was later promoted to a professor, at the Institute of Mathematics within the Faculty of Natural Sciences and Mathematics.

Over the years, my debt towards Professor Čupona was growing. He continuously monitored my work as a lecturer and my scientific research results, which he used to value as a reviewer, always evaluating objectively and precisely.

Unforgettable in my memories remain the after earthquake years in Skopje, not only for the very complex living and working conditions at the University. It was the period when me and my colleagues (fellow assistants) Aleksandar Samardžiski from FNSM and Naum Celakoski from the Faculty of Electrical Engineering, were admitted to the master (graduate) studies in mathematics at the University of Belgrade, as part-time students. We had to take exams in four general courses (Introduction to Algebraic Structures, Topology, Analysis 3, Linear Algebra) before choosing to take the Main Course Exam, all that followed by exams in two additional courses, different for each of us. Afterwards we were expected to select a research topic, then prepare a Master Thesis and publicly defend it. We were facing difficulties, since we had no teaching materials and appropriate instructions nor a corresponding literature for preparing all of the above. Professor Čupona was aware of the need for new academic staff and the arduous path of its creation. That is why he organized for us - postgraduates, so called mathematical rounds. We had all together meetings once a week, where we were presenting the contents learned and searching for problems to be solved. Afterwards we used to discuss, ask questions and sought answers, all that, guided by Professor Čupona. He actively participated in everything, as if he was about to take our exams himself. When we were not doing our best, as for example when struggling with some English translation of an entire book, he jokingly used to remark that we had to feel good and be satisfied, since we were about to become some of the most educated mathematicians in our Republic.

We have learned from Professor Čupona to reconsider before answering a question or before offering a solution to a problem (among other), and to analyse each topic from all aspects during its consideration.

I am grateful that I have such an honor to mark the 90th anniversary of the birth of Academician Professor Dr. Gjorgi Čupona, and to pay tribute on his Personality and Life Work. His achievements and dedication during his life are fundamental for the development of Macedonian Sciences.

I finish this homage text paid to the respected Professor, with a faith in better tomorrow in these times of storm, for all the good people in the world.

REMEMBERING GJORGJI ČUPONA

Smilka Zdravkovska, Ann Arbor, Michigan, USA

The first thought that comes to mind when I want to describe Čupona (that's how I referred to him after knowing him for several decades) is that he is someone who is always ready to help anyone, while being very modest, honest, direct, visionary and an idealist.

I met him when I entered high school in 1960, through his spouse Gorica Ilieva, my beloved mathematics teacher. She had a reputation for being very strict, but only because she made her students work hard and learn mathematics, while being very helpful. For this help she would enlist her husband as well, thus allowing him to scout potential future university students in mathematics and other fields.

Sometimes, though rarely, we would go to their apartment to get math problems; recently, others have told me that he had also helped them with math in the Čupona-Ilieva kitchen. This reminds me of the following. In 2002 I met by chance a very interesting and cultivated lady in her 90s, Anna Yona, with whom I spent several months talking. When she learned that I was a mathematician, she told me that when she was a schoolgirl, her parents asked her uncle to help her out with mathematics. It was thus that the young but already famous mathematician Fubini, as Anna told it, would occasionally get out on their balcony in Turin, pull his hair out and scream: "E una bestia!" I can't imagine Čupona ever doing that, as in my experience he was infinitely patient and modest.

All that work paid off; in 1964, the 9-member team representing Yugoslavia at the International Mathematics Olympiad in Moscow had two students of the Ilieva-Čupona couple: Viktor Urumov and myself. By the way, the team consisted of three members from Slovenia, four from Serbia, and two from Macedonia, and we are still occasionally in touch; our emails start with the greeting "Dragi Olimpijci." I can't help but think that Čupona would be pleased to know that the team representing Yugoslavia still persists.

Viktor and I went to study in Moscow, USSR, physics and mathematics, respectively, thanks to a Soviet scholarship as help to the city of Skopje after the devastating 1963 earthquake.

But the visionary nurturing by Čupona did not stop there. He established the Mathematical Institute with Numerical Center of the University of Skopje, and was its director. His idea was to create an interdisciplinary center at the university, where specialists in various fields of mathematics, computer science, economics, engineering, biology and others could work together. This was a very forward-looking vision at that time.

Čupona hired everybody in that Institute. First he hired Simeon Ivanov, Josif Hadzi-Pecov, and myself, all within a couple of months of each other in 1969. He secured one of the little prefabricated barracks near the Faculties of Technology and Engineering for offices and continued on his hiring spree of youngsters in all specialties. (There were so many young recruits by 1974, that four of us gave birth almost simultaneously.) I hope someone will write a history of this influential Institute, with substantiation; I looked for such a documented article on it, but did not find one, even though there are a few slightly contradictory accounts here and there.

Čupona was the first to teach topology in Macedonia, and he wrote really nice notes for this course. He collaborated with several topologists all over Yugoslavia, such as S. Mardešic, J. Vrabec, and others. But when I started working at the Mathematical Institute with Numerical Center, since the members of that Institute were supposed to teach at the University, Čupona passed on the teaching of topology to me, and generously shared his notes. He remained very active in the topology seminar, and he and I took care of the many talented students interested in the subject. It was only later that I realized just how much he had selflessly helped my students, especially after I left in 1979 for the USA. But they are the ones who already have or could write something more about that in the future.

I cannot write about Čupona without mentioning the lengthy hikes in the mountains, particularly Vodno and Matka, which he took with us younger colleagues, and where the conversation revolved not only around mathematics, but also politics, society, nature, life. He was very direct when he disagreed with his interlocutor, but he always radiated idealism, which I truly enjoyed.

I hope I have been able to illustrate, albeit very faintly, the sentence at the beginning of these memories.

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Review

PROFESSOR ČUPONA AND THE DEVELOPMENT OF COMPUTER SCIENCE IN MACEDONIA

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This paper reviews the contribution of Professor Čupona in development of Computer Science in Macedonia. The approach taken preserves history, and all events are listed with order of time. The approach also preserves the worldwide context of Computer Science in which development of Computer Science in Macedonia took place. The approach also looks for pioneering contribution of Macedonian Computer Science in the world and the role of Čupona in them. The paper emphasizes the role of Čupona in the first paper in Computer Science written in Macedonian language. Working on Čupona's contribution this research looks also for contributions of the Mathematical Institute with Numeric Center (MINC) and points out a worldwide contribution by this institution. The paper also points out Čupona's work as mathematician, on algebraic structures, and the influence of that work in education and research of Macedonian Computer Science.

Key words: Professor Čupona, development of computer science, Macedonia 1964–2017

INTRODUCTION

This invited paper is a personal view of the author of both the contribution of Professor Gjorgji Čupona and the development of Computer Science in Macedonia. The approach taken in this review is a research in history of Computer Science in Macedonia, preserving years when events happened. The approach also preserves the worldwide context of Computer Science in which development of Computer Science in Macedonia took place. The paper makes a research on Čupona's contribution as visionary, organizer, educator and inspirer of development of Computer Science, as science in Macedonia.

In the sequel we first observe the role of Dr. Čupona in mathematics competitions and nurturing the high school talent. Then we give information about a scientific event, appearance of a book on cybernetics by Glushkov, which inspired Dr. Čupona to determine the direction of development of Computer Science in Macedonia and initiate the first paper on Computer science written in Macedonian language. Then we point out the engagement of Dr. Čupona in building a scientific infrastructure, the Mathematics Institute with Numeric Center (MINC). Then we mention some pioneering results in development of computer science in Macedonia. The educational influence of Čupona as mathematician especially in the field of algebraic structures on education and research on Computer Science in Macedonia is also mentioned.

1960's: HIGH SCHOOL COMPETITIONS IN MATHEMATICS IN MACEDONIA. FIRST MEETING WITH DR. ČUPONA

Many high school students in 1960's met Dr. Čupona for the first time during preparations for mathematics competitions. Here we will mention a generation of mathematics competitors including Smile Markovski, Dimitar Altiparmakov, Tome Mickovski, Risto Ciconkov, Biljana Arsova, Eli Delidzakova, Gjorgi Josifovski, Stevo Božinovski, among many others. Here we will also mention the ”idols” Smilka Zdravkovska and Viktor Urumov, who achieved to participate at the International Mathematical Olympiad in Moscow. Three of the mentioned, same generation students, Dimitar, Smile and Stevo, participated at the federal competitions in Belgrade. The system of organized work with high school students included high school mathematics professors who worked with students preparing them for the competitions. One such example was Gorica Ilieva, the spouse of Professor Čupona. She had dear personality and knowledge to attract students towards mathematics. The author had a privilege and a pleasure to be one of those students.

Part of preparation for mathematics competitions were various mathematics schools organized by Dr. Čupona. During those activities the interested high school students were exposed to mathematics lectures beyond standard high school curriculum. For example, at that time the concept of a set was not in a regular high school mathematics curriculum. It was taught in those extracurricular activities, along with determinants, matrices, etc. The instructors at the mentioned mathematics schools were university professors, for example Naum Celakovski, Živko Madevski, Aleksandar Samardziski, Branko Trpenovski, among others.

The first meeting between this author and Dr. Čupona happened in 1965 when Gorica Ilieva sent this author to meet Dr. Čupona at the department of Natural Sciences and Mathematics. Not knowing the room number this author went to the library and asked a person how to find Professor Čupona. The person answered with room number and pointed a time later when Dr. Čupona will be there. This author went there at the given time, and a surprise, the person he met before in the library was actually Dr. Čupona. Many years later this author would learn from animal learning theory that according to Rescorla and Wagner, systems learn only when they are surprised. That is how a close collaboration started.

1960's: THE VISION OF PROFESSOR ČUPONA: DEVELOPMENT OF AN INSTITUTION RELATED TO COMPUTERS

During 1960's, worldwide and in Macedonia there was thinking how to guide a development related to computers and programming. Here we point out the vision of professors Gjorgi Čupona and Blagoj Popov who in 1966 formed a Mathematical Institute with Numerical Centre (MINC). It was established as an institution with special interest for the society. We recognize that it was the first institution in Macedonia which in its title had something related to computer science. The second institution who made such a movement was Electro-mechanical Faculty, which in 1972 established a Cathedra of Cybernetics, led by Professor Pane Vidinčev.

One may view that there is no significant difference of 6 years, but in this case there is. In those 6 years some events happened in the world and in Macedonia that influenced the path for the development of Computer Science in Macedonia.

1964: A WORLDWIDE: EVENT IN COMPUTER SCIENCE: A BOOK BY GLUSHKOV

At the time MINC was formed, an important event in science turned out to be influential in the development of Computer Science in Macedonia. In 1964 the book "Introduction to Cybernetics" by Viktor Glushkov appeared [1] offering a radical new view toward Cybernetics. While the previous view was related to control theory, the new book introduced Cybernetics as Computer Science, through theory of algorithms, languages, automata, and self-organizing systems. Here is the chapters overview:

1. The abstract theory of algorithms
2. Boolean functions and propositional calculus
3. Automata theory
4. Self-organizing systems
5. Electronic digital computers and programming
6. The predicate calculus and the automation of the processes of scientific discovery

Glushkov starts with the concept of algorithms in terms of machines by Post, Turing, and Markov, as well as knowledge on propositional calculus. The book then elaborates on automata theory. Basing on automata concept, the book has important contribution to the theory of self-organizing systems. The knowledge of computers and programming is necessary, and book covers the programming language ALGOL. It also covers the predicate calculus and automated reasoning, which is a topic of classical Artificial Intelligence. It gives algebraic treatment of most of the topics covered. The book brought ideas from other researchers in the field of learning for pattern recognition, for example from Rosenblatt and his neural network named Perceptron [2, 3] and Selfridge and his pattern recognizing architecture named Pandemonium [4]. It contained also the newest research of Glushkov himself on abstract automata and self organizing systems [5].

1967 AND 1969: THE CONTEXT: ABSTRACT AUTOMATA WORLDWIDE

To consider the context of development of Computer Science in Macedonia, we will now observe the worldwide development of automata theory through two significant books published in 1967 and 1969.

Although theories of abstract automata were developed in 1950' [e.g. 6], after the Glushkov's treatment of automata theory, two other books relevant to Computer Science in Macedonia, were published.

In 1967 appeared a book by Marvin Minsky with a title "Computation: Finite and Infinite Machines" [7]. To see an influential view on automata theory at that time, here we list its chapters.

Part I: Finite state machines

1. Physical machines and their abstract counterparts.
2. Finite state machines
3.Naural networks: automata made up of parts
4. Memories of events in finite state machnes

Part II. Infinite machnes

5. Computability, effective procedures, and algorithms
6. Turing machines
7. Universal Turing machines
8. Limitations of effective computability: some problems not solvable by instruction-obeying machiens
9. Computable real numbers
10. The relation between Turing machines and recursive functions
11. Models similar to digital computers

Part III. Symbol manipulation systems and computability

12. Symbol manipulation system by Post
13. Post's normal form theorem
14. Very simple bases for computability

In 1969 appeared a book by Michael Arbib entitled "Theories of Abstract Automata" [8]. The list of chapters is

I Background

1. An overview of automata theory
2. Algebraic background

II An introduvtion to automata theory

3. Finite automata
4. Turing machines and effective computation
5. Post system and context-free languages

III Selected topics

6. Partial recursive functions
7. Complexity of computation
8. Algebraic decomositon theory
9. Stochastic automata
10. Machiners which copute and construct

1968: THE BOOK BY GLUSHKOV AS AN INSPIRATION FOR PROFESSOR ČUPONA

At this point we observe a context of development of Computer Science in the world in the field of abstract automata, and in Macedonia we observe an effort of Dr. Čupona to develop something related to mathematics and computers. Here comes the book of Glushkov.

The Glushkov's book was translated in several languages, for example in USA it was translated in 1966 [9]. In Yugoslavia it was translated by Rajko Tomovic and Momcilo Uscumlic in 1967 [10]. The book was presented at the Book Fair in Skopje. At that time the Fair was in old location on the left bank of river Vardar. Important coincidence was that at that Book Fair the Glushkov's book was purchased by Dr. Čupona, and, independently, by the author of this paper. The cover of this book is shown in Figure 1.

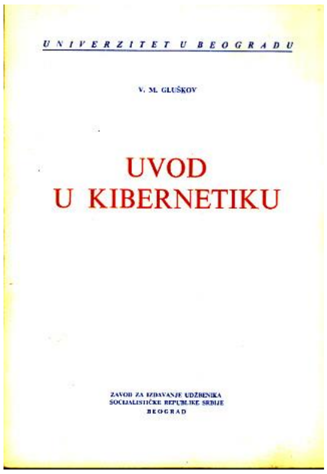


Figure 1. The book which inspired Professor Čupona to guide the development of Computer Science in Macedonia

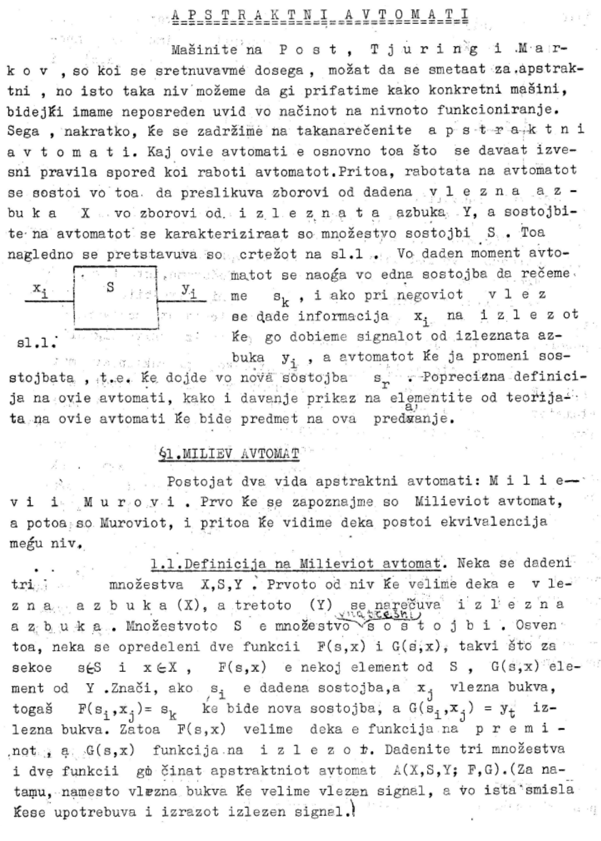
The visionary Dr. Čupona realized that the abstract automata theory is the way to go for development of a computer science. His spouse Gorica Ilieva was mathematics teacher of the author of this text. The year 1968 was the year when this author had to choose a topic of his high school (matural) thesis. One day Gorica Ilieva brought the book by Glushkov and showed it to this author and proposed that the chapter on abstract automata be the Matural Thesis in mathematics. The author of this text said that he is accepting the proposal, but he does not need the book, because he had already purchased it at the Book Fair. That way Dr. Čupona learned about the coincidence of purchasing the same book with this high school student.

So, a work started of studying the automata theory by a high school student, and Glushkov became his teacher. During this period Dr. Čupona was interested to see the progress of the work. While the supervisor of the Matural Thesis was Professor Gorica Ilieva, many discussions with Dr Čupona were on the topic.

Having a vision of the development of mathematics, computer science and cybernetics, Dr. Čupona decided to organize a seminar on cybernetics motivated by the book of Gluskov. In 1968 professor Čupona organized the First Seminar on Cybernetics in Macedonia. He used the first three chapters of the Glushkov's book to be guidelines for the seminar.

Nine lectures were presented by university professors at the seminar including Naum Celakoski, Branko Trpenovski, Živko Madevski, Aleksandar Samardziski, and Gjorgji Čupona. He also invited lecturers related to cybernetics from standpoint of control theory (Pane Vidinčev) as well as from standpoint of biology (Lav Lozinski).

At that time there was no university instructor in Macedonia working on abstract automata. But there was a high school student who already wrote his high school thesis entitled Abstract Automata. Considering all that, Professor Čupona decided that the 10th lecture of the seminar will be on abstract automata and will be delivered by the high school student who already had knowledge of the subject. The lecture was written by this author as a handout paper, reproduced on schapyrograph (a copying technology of that time), The paper was edited by Dr. Čupona. It was distributed among the participants of the seminar in May 1968. The first page of the paper [11] is shown in Figure 2.



A P S T R A K T N I A V T O M A T I

Mašinite na Post, Tjuring i Markov, so koi se sretnuvavme dosega, možat da se smetaat za apstraktni, no isto taka niv možeme da gi prifatime kako konkretni mašini, bidejḱi imame neposreden uvid vo načinot na nivnoto funkcioniranje. Sega, nakratko, ḱe se zadržime na takanarečenite apstraktni avtomati. Kaj ovie avtomati e osnovno toa što se davaat izvesni pravila spored koi raboti avtomatot. Pritoa, rabotata na avtomatot se sostoi vo toa da preslikuva zborovi od dadena vlezna azbuka X vo zborovi od izleznata azbuka Y, a sostojbite na avtomatot se karakteriziraat so množestvo sostojbi S. Toa nagledno se pretstavuva so crtežot na sl.1. Vo daden moment avtomatot se naoǵa vo edna sostojba da rečeme s_k, i ako pri negoviot vlez se dade informacija x_i na izlezot ḱe go dobieme signalot od izleznata azbuka y_i, a avtomatot ḱe ja promeni sostojbata, t.e. ḱe dojde vo nova sostojba s_r. Popreciznа definicija na ovie avtomati, kako i davanje prikaz na elementite od teorijata na ovie avtomati ḱe bide predmet na ova predavanje.

§1. MILIEV AVTOMAT

Postojat dva vida apstraktni avtomati: Milievi i Murovi. Prvo ḱe se zapoznajme so Milieviot avtomat, a potoa so Muroviot, i pritoa ḱe vidime deka postoi ekvivalencija meǵu niv.

1.1. Definicija na Milieviot avtomat. Neka se dadeni tri množestva X,S,Y. Prvoto od niv ḱe velime deka e vlezna azbuka (X), a tretoto (Y) se narečuva izlezna azbuka. Množestvoto S e množestvo vnatrešni sostojbi. Osven toa, neka se opredeleni dve funkcii F(s,x) i G(s,x), takvi što za sekoe $s \in S$ i $x \in X$, F(s,x) e nekoj element od S, G(s,x) element od Y. Znači, ako s_i e dadena sostojba, a x_j vlezna bukva, togaš $F(s_i,x_j) = s_k$ ḱe bide nova sostojba, a $G(s_i,x_j) = y_t$ izlezna bukva. Zatoa F(s,x) velime deka e funkcija na preminot, a G(s,x) funkcija na izlezot. Dadenite tri množestva i dve funkcii go činat apstraktniot avtomat A(X,S,Y; F,G). (Za natamu, namesto vlezna bukva ḱe velime vlezen signal, a vo ista smisla ḱe se upotrebuva i izrazot izlezen signal.)

Figure 2. Beginning of the first paper in Computer Science in Macedonia written by Macedonian language. It contains a handwritten editing note by Professor Čupona.

We would point out that a review of applicability of automata theory is given in a previous paper devoted to Professor Čupona in 2010 [12].

THE IMPORTANCE OF THE 1968 PAPER

This paper shows the vision of Professor Čupona toward the development of mathematics, computer science, and cybernetics in Macedonia, that it should include automata theory and related topics. This paper marks the start of Computer Science in Macedonia.

Dr. Čupona introduced abstract automata in Macedonia in 1968, between the 1967 book of Minsky and 1969 book of Arbib. The Čupona's effort was on time, enabling the Macedonian science to catch up with development of Computer Science in the world.

Dr. Čupona inspired a high school student to write the first paper on Computer Science in Macedonian language. He created a student who was well educated and competing in mathematics, and now he already studied automata theory and related topics from a teacher such as Viktor Glushkov. Reading the book, the student viewed the next chapter after automata theory, which was on perceptrons, machine learning, and related topics in Artificial Intelligence. In addition to that, Dr. Čupona brought the student to the home of Lav Lozinski. The three of them, in Professor Lozinski's room, filled with piles of books on the floor, discussed about application of mathematics in biology.

The formal organizer of the 1968 seminar was MINC. It was one of contributions of this institution to development of Computer Science in Macedonia.

1969: A WORLDWIDE EVENT. THE BOOK OF MINSKY AND PAPERT

Two years after his book on abstract machines, Minsky decided to give contribution to pattern recognizing learning machines, and in 1969 a book appeared by Marvin Minsky and Seymour Papert named "Perceptrons" [13]. This book points some limitations of perceptrons. By many AI reserchers it was interpreted as neural network research is not promising. Because Minsky was very influential name in Computer science, the National Science Foundation (NSF) of USA stopped financing the artificial neural networks research.

In that worldwide context, after the 1968 lecture and paper, this author continued his study at the Electrical Engineering department (ETF) of University of Zagreb. At ETF Zagreb, this author looked for opportunity to work on chapter IV of the Glushkov's book, self-orgnizing systems. In 1971 University of Zagreb opened a competition for a student scientific work, named 1st of May Prize. This authot went to Professor Ante Santic who was the only professor having the word perceptrons in his syllabi. Dr Santic gave a support, and the paper was written [14], entered the competition, and was awarded by the university Rector, Professor Ivan Supek. More importantly, perceptrons and machine learning were now already studied by this author. Also, the first software was written, in Fortran language, to simulate the learning process of a perceotrion in recognizing patterns on a binary retina.

Regarding the book of Minsky and Papert, this author read it at that time, 1971, in Zagreb, in Russian translation. From reading the book, for this author it did not seem evident that the neural networks are not a promising direction, and he decided that he will continue his research on neural networks and related topics.

In the meantime, in 1971 ETF in Zagreb opened the first undergraduate computer science program in Yugoslavia. This author was the first generation students of that program.

A series of works followed on neural networks. An undergraduate seminar work on digital integrated circuit (DTL technology) simulating a neural network for conditioned reflex [15], was the first work on simulation a neural network in hardware. It was used later in a textbook by Professor Santic [16, Fig. 1, 28]. The undegraduate Diploma Thesis was on simulation of neural elements with both impulse (astable multivibrator) and digital electronics [17]. The Master's Thesis was on software simulation of perceptroms for pattern recognition. [18].

So, despite the view of Minsky and Papert [13], this author continued working on neural networks believing that neural networks approach is very promising.

1974–76: MINC: APPLICATIVE COLLABORATION WITH OTHER INSTITUTIONS

In 1974 the author of this text joined the Mathematical Institute with Numerical Centre (MINC).

Since 1968 this institution has grown significantly. The directors of MINC were B. Popov (1966–1969), G. Čupona (1969–1973), I. Šapkarev (1973–1975), and Ž. Madevski (1975–1977).

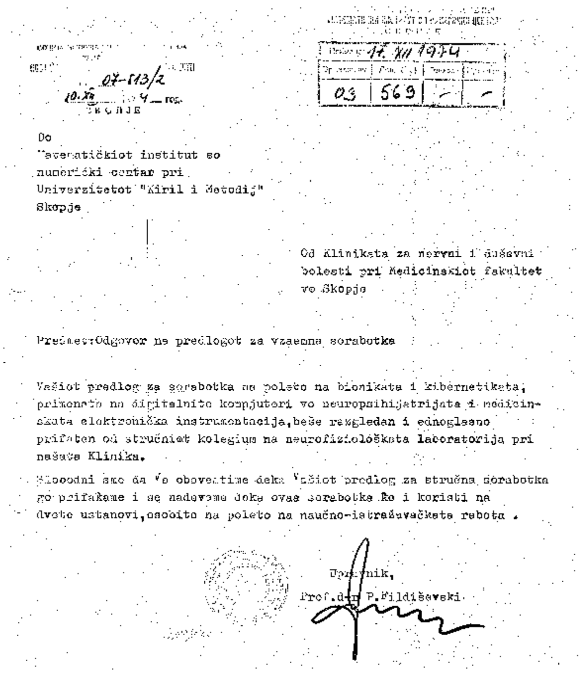
By 1976 MINC employed 16 people from various disciplines, including mathematicians, electrical engineers, computer scientists, and economists. Here is the list of scientists engaged by MINC with years of their employment:

S. Zdravkovska, S. Ivanov, J. Hadzi-Pecov (1969), T. Šurležanovska (1970), R. Šokarovski, D. Korobar Tanevska, N. Sekulovska, Lj. Stefanova (1971), V. Naumovska , V. Kusakatov (1972), D. Mihajlov, D. Nikolic, M. Kon-Popovska, V. Naumovska (1973), S. Božinovski, M. Simova (1974), D. Dimovski (1976).

The Institute also had external collaborators, people employed in other institutions.

MINC engaged in two directions, applicative and scientific. Because MINC was a creation of Dr. Čupona, here we will mention the applicative and scientific work undertaken within MINC by this author.

The first applicative work was a collaboration between MINC and Clinic of Neuropsychiatry. Figure 3 shows the collaboration agreement.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Do
Matematičkiot institut so
numerički centar pri
Univerzitetot "Kiril i Metodij"
Skopje

Od Klinikata za nervni i duševni
bolesti pri Medicinskiot fakultet
vo Skopje

Predmet:Odgovor na predlogot za vzaemna sorabotka

Vašiot predlog za sorabotka na poleto na bionikata i kibernetikata, primenata na digitalnite kompjuteri vo neuropsihijatrijata i medicinskata elektronička instrumentacija,beše razgledan i ednoglasno prifaten od stručniot kolegium na neurofiziološkata laboratorija pri našata Klinika.

Slobodni sme da Ve obvestime deka Vašiot predlog za stručna sorabotka go prifakame i se nadevame deka ovaa sorabotka ke i koristi na dvete ustanovi,osobito na poleto na naučno-istražuvačkata rabota .

Upravnik,
Prof.d-r P.Fildiševski

Figure 3. The collaboration agreement between MINC and Clinic of Neuropsychiatry, 1974

The collaboration was established by letters from directors of both institutions, by Professor Ilija Šapkarev for MINC and Professor Petar Fildisevski for the Neuropsychiatric Clinic. It was the first applicative formal collaboration of MINC with another institution. The collaboration with this clinic enabled working with section for biosignals, especially EEG. The first teacher to this author about reading an EEG was Dr. Liljana Dzambaska, but collaboration was carried out also with Dr. Aleksandar Naumovski and Dr. Vera Ivanova.

Next collaboration was established with Zavod for Mental Health of Children and Youth. Collaboration was established on the project for growth and development of the children in the first three years. It was a project supported by USA, and project representative was Professor Robert Reed, from Department of Biostatistics, Harvard School of Public Health. Two people were engaged from MINC, the author of this text and Simova Marija. They worked on processing the Binet-Simon scale. Collaborators from the other institution were Ruzica Keramitcieva and Sineva Joveva.

Another collaboration was established in 1976 with Institute of Physiology led by director Professor Vanco Kovacev. It was for the project on physical and functional characteristics of the population in Macedonia. A database was built containing 150,000 entries. The author of this paper worked together with Margita Kon-Popovska. This project brought the first payment to MINC for its applicative engagement.

1976: MINC: THE FIRST MACEDONIAN WORLDWIDE ACHIEVEMENT IN COMPUTER SCIENCE

Although applicative work was encouraged by MINC, the scientific work was its primary focus. Three papers of this author are mentioned here:

In 1975 a paper was published in the Review of Psychology which was printed in Zagreb, and was about abstract automata and neural networks [19]. The pioneering scope of this paper is local, it is the first Artificial Intelligence journal paper by a Macedonian author.

The second paper was a local paper entitled "An approach toward threshold elements and formal neurons" [20] and was actually a presentation for MINC about the research in neural networks carried out at MINC. The presentation advisor was Smilka Zdravkovska.

The third was a pioneering paper published in 1976 as a conference paper. It was entitled "Influence of pattern similarity and transfer of learning on the training the base perceptron B2" [21]. Many years later it was realized worldwide that transfer learning is important concept of Machine Learning. The second paper of the topic was published in USA in 1991 fifteen years after this pioneering paper [22]. This paper written by an employee of MINC in 1976 is now cited in historical part of the Wikipedia topic on transfer learning [23, 24]. This achievement of MINC is due to engagement of Dr Čupona and the 1968 paper.

It is important to mention a 1976 work of Professor Čupona. Being great supporter of development of MINC and Computer Science in Macedonia, he also remained primarily professor of mathematics. In 1976 he published his textbook "Algebraic Structures and Real Numbers" [25]. Later it will become relevant to education of Computer Science in Macedonia.

1977‒1979: ČUPONA'S WORK ON COMPUTER SCIENCE EDUCATION THROUGH MATHEMATICAL SCHOOLS

Professor Čupona was engaged in Mathematical Schools for education of students interested in mathematical competitions through mathematical summer schools. After 1968 he realized that computer science education should be applied at level of the summer mathematical schools. At that time, at the end of 1976, this author joined the Cathedra of Cybernetics of the Electro-mechanical Faculty. He started teaching as teaching assistant the subject of digital computers following the book of Branko Souček, who was his professor of computers and processes at the University of Zagreb.

In 1977 Professor Čupona was organizing a mathematical school in Ohrid, and he asked this author to carry out a mathematical school providing deeper knowledge than the standard Fortran programming. A 44-page booklet was prepared [26] to support teaching the relation between computer hardware and software as well as programming at the machine level.

1979–1981: FOLLOWING THE VISION OF DR. ČUPONA, JOINING A MILITARY RESEACH ON NEURAL NETWORKS IN USA

In 1979 two events happened which highly influenced development of Computer Science in Macedonia and in the world.

First, the author of this text applied for a Fulbright scholarship. He met Dr. Arbib at a conference in Bled and asked him if he is interested in neural networks research. Arbib said that the author should apply to his institution, University of Massachusetts at Amherst.

The second event was related to the context created by the book of Minsky and Papert. In 1979 there was no federal funding of artificial neural networks research. However, the Air Force base in Dayton, Ohio, decided that artificial neural networks research is needed, and decided to finance itself such a research. They opened a project at the University of Massachusetts at Amherst in 1979 at the time the author of this text asked Arbib to work on neural networks. Arbib recommended contact with Professor Nico Spinelli, who was the leader of the group, named Adaptive Networks (ANW) group carrying out the project. So, the author of this paper was accepted to work on a military funded project on neural networks.

Observing this event, we can see that the vision of Dr. Čupona created a student who is now proficient in theory and programming of neural networks and is part of USA military project on that topic. So, the firm belief of this author in neural networks, rather than the common belief that those are "weak methods" in Artificial Intelligence, paid off. The vision of Dr. Čupona now reached the worldwide science.

Having both theoretical and programming knowledge of neural networks this author was able to produce significant pioneering results. Some of them are introduction of the concept of self learning (besides supervised and reinforcement learning) and, introduction of emotion in neural network learning, introduction of genetics in neural networks learning, the solution of the problem of reinforcement learning with delayed rewards, among others [27–30]. Those are results of the knowledge gained from the Čupona's 1968 vision. It should be mentioned that direction and the challenges related to reinforcement learning was defined by Adaptive Networks (ANW) group which carried out the project (in 1981: Spinelli, Arbib, Barto, Sutton, Anderson, Porterfield, Bozinovski). However, the self-learning research direction, based on emotion and genetics, was determined by the author of this text.

In 1986 a book appeared by David Rumelhart, Jay McClelland, and the parallel distributed processing group entitled "Parallel Distributed Processing" [31]. It basically said that the neural networks research is indeed promising. So, NSF started again financing such research, and neural networks research become a mainstream in Artificial Intelligence.

1982–1989: MORE PIONEERING RESULTS BY THE MACEDONIAN COMPUTER SCIENCE

During 1982–1988 the author stayed in intensive contact with Dr Čupona. Each year since high school he visited his home for the birthday of Professor Gorica Ilieva. Many times, this author and Dr. Čupona hiked on mountain Vodno. One was night hiking over Vodno to the water spring above village Sopiste. A hike took place on mountain Skopje's Montenegro, Dr. Čupona was always informed and interested in the work of this author.

Several worldwide pioneering results happened in this period and here will be mentioned two of them.

In 1986 the first speech-controlled robot was built in Macedonia [32, 33]. A robot was controlled by the following commands: "trgni", "nazad", "levo" desno" and "stoj". Because the robot was created in Macedonia, and understood only the Macedonian language, we may say that the first in the world robot whose movement was controlled by speech commands was a Macedonian. It is a possibly a strange sentence but in a sense it is true.

The second worldwide pioneering result is the first control of a robot using signals emanating from a human brain. It happened in 1988 [34] and it solved the long lasting problem, the engineering solution of the science fiction belief in psychokinesis, movement of a physical object using only energy emanating from a human brain.

The knowledge used to approach this scientific challenge was partly influenced from automata theory initiated by Dr. Čupona, the mathematical methods in biology influenced by inspiration from Dr. Čupona and Professor Lozinski in 1968, and EEG knowledge obtained in collaboration between MINC and Neuropsychiatric clinic, mentioned before.

Figure 4 shows the abstract automata graph (Moore type) used in the 1988 design of control of robot using EEG signals.

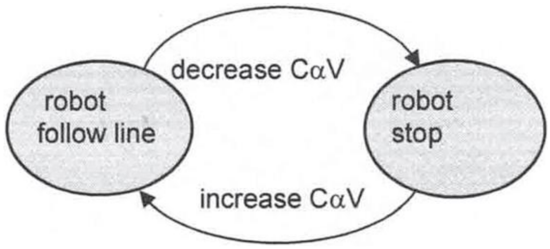

Figure 4. Abstract automaton (Moore type) used in design of the pioneering control of robot movement using EEG signals

As Figure 4 shows, initial (default) robot behavior is following a line on the floor using own artificial intelligence. If a human increases his alpha rhythm amplitude (Contingent alpha variation (CαV)) the robot stops. If CαV is decreased the robot continues its default behavior.

It should be noted that the next result of controlling a robot using EEG signals was reported in USA [35], 11 years after the Macedonian result.

This 1988 event is currently recognized as a historic event in science related to bioelectric signals. The Royal Society created a timeline chart placing this event as historical achievement [36]. Figure 5 shows part of that history chart.

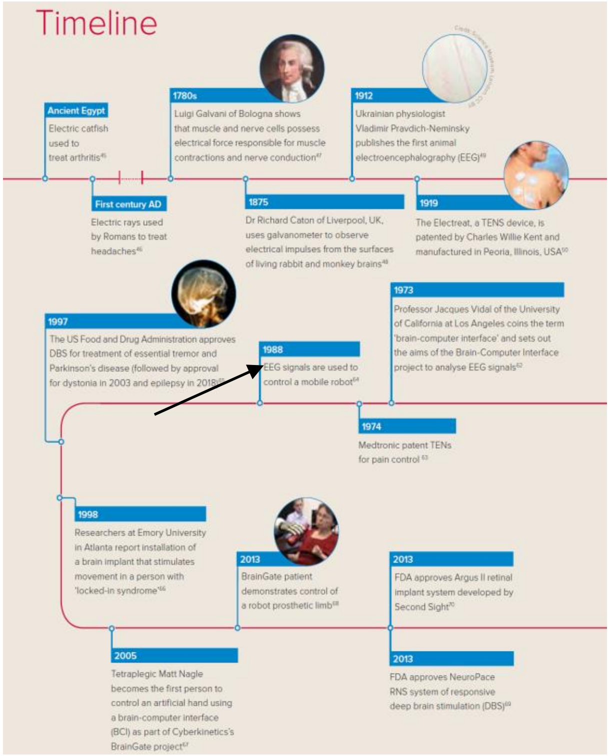

Figure 5. A Royal Society history chart starting from ancient Egypt, pointing out the 1988 historic event

The chart shown in Figure 5 has another page, to the right, not shown here. Note that the chart on Figure 5 should be read left-to-right and then right-to-left. It should be noted that the Royal Society is the oldest scientific academy in continuous existence, and in 1687 they published Newton's Philosophiae Naturalis Principia Mathematica.

1986–1998: ČUPONA AND HIS CARE FOR THE PEOPLE WORKING ON COMPUTER SCIENCE IN THE INSTITUTE OF MATHEMATICS AND INFORMATICS

Institute of Mathematics and Informatics continued the work of MINC. It was a part of Department of Natural Sciences and Mathematics. Dr. Čupona took care of the people employed in this institution. He asked the author of this text to be a Mentor of the PhD Thesis of G. Jovančevski. That way the collaboration continued with Institute of Mathematics and Informatics.

This collaboration produced a work on using a neural network in tuning an operating system [37]. Also it produced a work related to graphical representation of neural network learning in the teaching space and viewing neural network teaching as an integer programming problem. This part of research was carried out with help of Professor Dimitra Karčicka [38, 39]. All this work and collaboration was initiated by Dr. Čupona.

1994–1998: INFLUENCE OF ČUPONA ON MATHEMATICAL EDUCATION OF STUDENTS OF COMPUTER SCIENCE IN MACEDONIA

This author was inspired by the work of Dr. Čupona and his work on algebraic structures [25]. Figure 6 shows cover page of Čupona's book on algebraic structures.

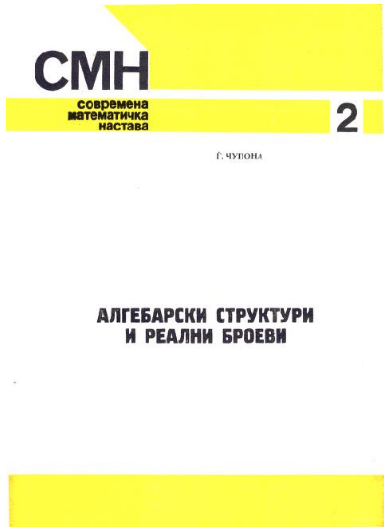

Figure 6. The cover of the 1976 Čupona's book on algebraic structures

Here we list the chapters of the book:

1. Elements of set theory
2. Grupoids. Operations with natural numbers
3. Integers
4. Congruences and isomorphisms. Rational numbers
5. Ordered fields. Real numbers

Influenced by the work on Dr. Čupona on algebraic structures the author of this text provided to his students of Computer Science at the Electrical Engineering Faculty some background on algebraic structures. Two of his textbooks explicitly contained a mathematical Appendix covering the topic.

One of the books was entitled "Operating Systems and Systems Software I: Von-Neumann Computers and Monoprocessing Operating Systems" [40]. The cover page is given in Figure 7.

Figure 7. A 1994 Computer Science book having an Appendix covering algebraic structures influenced by the work of Dr. Čupona.

Here is the list of chapters of the mathematical appendix of this book:

Mathematical basis of operating system DOS and UNIX

1. Files
11. Congregations, collections, sets, populations
1.2. Orderings
1.3. Tree-like orderings. rt strings
1.4. Files, file bases, file spaces, file addresses. Complete file names
2. Commands
2.1. Functions. Compositions
2.2. File functions
2.3 Assumed domains and codomains
2.4. Composition of file functions
2,4, Reporting functions
2.5. Commands
3. Operating systems DOS/UNIX
3.1. File spaces of DOS/UNIX
3.2. Commands of DOS/UNIX

It can be seen that this appendix not only gives background of algebraic structures, but it also uses them to model file structures. In addition, this Appendix has a view that mathematics should start with a concept of congregations (in the book the author used the Macedonian word "zbirstina") because a set has certain rules of formation.

The second Computer Science book containing an algebraic structure Appendix was the book entitled "Robotics and Intelligent Manufacturing Systems" [41]. Its mathematical Appendix contains the following chapters:

1. Non-structured: congregations, collections, sets, populations
2. Primitive structure. Full (Cartesian) product
3. Relations and graphs.
3.1. Transitive relations and their taxonomy
4. From structures to algebras
4.1 Partially ordered sets
4.1.1 Graphical representations
5. Supremum and infimum
6. Lattice structures
7. Algebraic structures
8. Boolean algebras

2005: LAST MEETING WITH PROFESSOR ČUPONA

After 2001 this author joined the Mathematics and Computer Science Department of South Carolina State University. The meetings with Dr. Čupona become rare. The last one was at the funeral of Professor Branko Trpenovski in May 2005. It was just a short, cordial communication, in presence of other people. There was no time for mountain hiking.

2014–2017: ČUPONA'S INFLUENCE IN A MATHEMATICAL THINKING: FROM HIGH SCHOOL COMPETITIONS TO TURING MACHINES AND INFINITE SERIES OF INTEGERS

Here we will mention a mathematical reasoning influenced by the high school competition, mentioned at the beginning chapter of this paper. The reasoning starts with the mathematical term we used very much in mathematics competitions, a^2-b^2. If we consider integers, then for two consecutive integers a = n+1, b = n, where n is even, we obtain the result $n + (n+1) = - n^2 + (n+1)^2$ [42]. For example $0+1 = -0^2+1^2$, $2+3 = -2^2 + 3^2$, etc. Summing those equations step by step to infinity we have

$$1+2+3+4+5+... = 1^2-2^2+3^2-4^2+5^2 - ... \quad (1)$$

The left side of equation is known as Euler-Riemann zeta function for argument -1, ζ(-1), and the right side is Dirichlet eta function for argument -2, η(-2). The equation (1) can graphically be illustrated as in Figure 8.

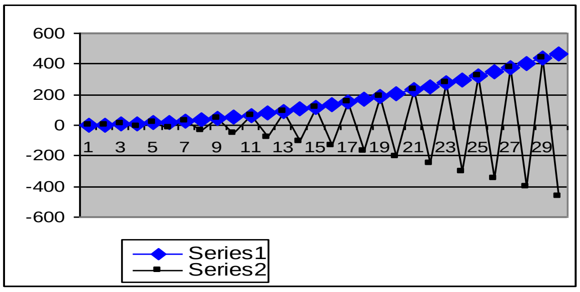

Figure 8. The relation between ζ(-1) (Series1) and η(-2) (Series2)

It can be seen from Figure 8 that the function ζ(-1) is the above envelope of the oscillating function η(-2). That can be described for example by

$$\eta_n(-2) = \zeta_n(-1)\cos(n-1)\pi \qquad (2)$$

where n denotes the n-th partial sum and n goes to infinity. In a Computer Science paper published on this subject [43] equation (1) is related to integers, Turing machines, and infinity.

This work shows the influence and legacy of Dr. Čupona through mathematical competitions, automata and Turing machines, algebraic structures, and integers, which lasts till recent days.

DISCUSSION

This paper takes a historical review, pointing time periods of the development of Computer Science in Macedonia, where Professor Čupona has an essential role. The paper shows the competitive work done in Macedonia in the field of Computer Science and relates it to the worldwide results. Professor Čupona significantly contributed to this cultural competition.

Following the vision of Dr. Čupona, Macedonian Computer Science was able to produce several pioneering results. Here mentioned is the first work on transfer learning in neural networks (1976) in which case the second such work was carried out 15 years later, in 1991, in USA. It is pointed out that this result was achieved by Mathematical Institute with Numeric Center (MINC) which was a creation of Dr. Čupona. Another work mentioned here is the first control of a robot using EEG signals in 1988, in which case the second such work was carried out 11 years later, in 1999, in USA.

CONCLUSION

Professor Čupona made a very significant impact of the development of Computer Science in Macedonia at several levels:

1) As visionary, he has seen that Computer Science should follow the way of algorithms, abstract automata, Turing machines, and related topics.

2) As organizer of Computer Science infrastructure, he formed the Mathematical Institute with Numeric Center in 1966.

3) He was active in working with high school students and mathematical competitions.

4) He was significant supporter of the Institute of Mathematics and Informatics.

5) He created conditions that allowed production of worldwide pioneering results, for example the 1976 result produced by MINC on transfer learning in neural networks.

6) He took care of development of computer science oriented people at Institute of Mathematics and Informatics.

7) He provided educational background on algebraic structures which was followed by many mathematicians as well as computer scientists.

8) He inspired many of his followers to think in terms of algebraic structures, and related topics.

REFERENCES

[1] V Glushkov, *Vvedenie v Kibernetiku,* Izdanie Akademii Nauk Ukrainskoi SSR, Kiev, 1964.

[2] F. Rosenblatt, The perceptron: A probabilistic model for information storage and organization in the brain, *Psychological Review* **65** (1958), pp. 386–408.

[3] F. Rosenblatt, *Principles of Neurodynamics,* Spartan Book, 1962.

[4] O. Selfridge, Pandemonium: A paradigm for learning, *Proc Symposium on Mechanization of Thought Process,* London, Her Majesty Stationary Office, (1958) pp. 523–526.

[5] V. Glushkov, Self organizing systems and abstract theory of automata. (In Russian) *Zhurnal Vychislitelnoi Matematiki i Matematicheskoi Fiziki* **3** (1962), pp. 459–466.

[6] C. Shannon, J. McCarthy (Eds.), *Automata Studies.* Princeton University Press, 1956.

[7] M. Minsky, *Computation: Finite and Infinite Machines,* Prentice-Hall, 1967.

[8] M. Arbib, *Theories of Abstract Automata*, Prentice-Hall, 1969.

[9] V. Glushkov, *Introduction to Cybernetics,* Academic Press, 1966.

[10] V. Gluskov, *Uvod u Kibernetiku*, Univerzitet u Beogradu, Beograd, 1967.

[11] S. Božinovski, Abstract automata (edited by Gj. Čupona) (In Macedonian) Topic 10, *First Seminar on Cybernetics,* Mathematical Institute with Numerical Centre, Skopje, 1968.

[12] S. Bozinovski, Professor Čupona and the first lecture on abstract automata in Macedonia, *Mathematica Macedonica* **8** (2010), pp. 79–94.

[13] M. Minsky, S. Papert, *Perceptrons: An Introduction to Computational Geometry*, MIT Press, 1969.

[14] S. Božinovski, Perceptrons problem: Pattern recognition teaching (In Croatian), Paper for *First-of-May student scientific competition,* University of Zagreb, 1971.

[15] S. Božinovski, V. Mesaric, An example of digital simulation of a formal neuron: Simulation of Pav-

lov's conditioned reflex using integrated circuits. (In Croatian) Seminar work, *Course of Biomedical Electronics*, Electrical Engineering Department, Zagreb, 1972.

[16] A. Santic, *Biomedical Electronics* (In Croatian), Zagreb, Skolska Knjiga, 1995.

[17] S. Božinovski, Electronic Models of Neurons (In Croatian) *Undergraduate Diploma Thesis*, Electrical Engineering Department, University of Zagreb, 1972.

[18] S. Božinovski, Perceptons and possibility of training for pattern recognition. (In Croatian) *Master's Thesis*, Electrical Engineering Department, University of Zagreb, 1974.

[19] S. Božinovski, Introduction to Psychocybernetics. (In Croatian) *Revija za Psihologiju*, **5** (1975), pp. 115–128.

[20] S. Božinovski, An approach toward threshold elements and formal neurons, (In Macedonian), Mathematical Institute with Numerical Centre, Skopje, 1976.

[21] S. Božinovski, A. Fulgosi, The influence of similarity and the transfer learning on the training of base perceptron B2 (In Croatian), *Proceedings of the Symposium Informatica*, 3-12-1-5, Bled, 1976.

[22] L. Pratt, J. Mostow, C. Kamm, Direct transfer of learned information among neural networks, *Proceedings of the Ninth National Conference on Artificial Intelligence* (AAAI-91), Anaheim, CA, (1991), pp. 584–589,

[23] S. Bozinovski, Reminder of the first paper on transfer learning in neural networks, 1976. *Informatica* **44**, (2020), pp. 291–302.

[24] Wikipedia >Transfer Learning > History https://en.wikipedia.org/wiki/Transfer_learning (October 2020).

[25] G. Čupona, *Algebraic Structures and Real Numbers* (In Macedonian) Prosvetno Delo, 1976.

[26] S. Božinovski, Elements of the design of hardware and software of digital minicomputers. (In Macedonian) Mathematical School, Mathematical Institute with Numeric Center, Ohrid, 1977.

[27] S. Bozinovski, Crossbar Adaptive Array, *Adaptive Networks Group*, Computer and Information Science Department, University of Massachusetts at Amherst, November 25, 1981.

[28] S. Bozinovski. A self-learning system using secondary reinforcement. *Abstracts of the Sixth European Meeting on Cybernetics and Systems Research* - 1982, Abstract 87, Austrian Society of Cybernetic Studies, Vienna, 1982.

[29] S. Bozinovski, A self-learning system using secondary reinforcement, in: *Cybernetics and Systems Research*, R. Trappl (ed.), North-Holland, 1982, pp. 397–402.

[30] S. Bozinovski. Crossbar Adaptive Array: The first connectionist network that solved the delayed reinforcement learning problem, in *Artificial Neural Nets and Genetic Algorithms*, A. Dobnikar, N. Steele, D. Pearson, R. Albrecht (eds.), Springer Verlag, 1999, pp.320–325.

[31] D. Rumelhart, J. McClelland, and PDP Group, *Parallel Distributed Processing. Exploration in the Microstructure of Cognition*, MIT Press, 1986.

[32] G. Grujovski, Realization of a system for recognition of isolated words, and control of a mobile robot with speech commands (In Macedonian), *Technical Report*, *Project Adaptive Industrial Robots*, 1986.

[33] G. Grujovski, S, Božinovski. Realization of a system for speech control of a mobile robot (In Macedonian), *Proceedings of the Yugoslav symposium of applied robotics and flexible automation*, Novi Sad, (1989), pp. 227–235.

[34] S. Bozinovski, M. Sestakov, L. Bozinovska, Using EEG alpha rhythm to control a mobile robot", in: *Proc 10th Annual Conference of the IEEE Engineering in Medicine and Biology Society*, G. Harris and C. Walker (eds.), New Orleans, LA, vol. **10**, track 17: Biorobotics, (1988), pp. 1515–1516.

[35] J. Chapin, K. Moxon, R. Markowitz, M. Nicolelis. Real-time control of a robot arm using simultaneously recorded neurons in the motor cortex. Nature Neuroscience **2** (1999), pp. 664–670.

[36] The Royal Society. iHuman: blurring lines between mind and machine, 2019, ISBN: 978-1-78252-420-5.

[37] S. Božinovski, G. Jovančevski. Operating systems tuning: An approach using trainable systems of Artificial Intelligence. (In Macedonian) *Proceedings ETAN*, Bled, **XII** (1987), pp. 231–236.

[38] G. Jovančevski, S. Božinovski. The problem of neural network training as a task of linear integer programming (in Macedonian), *Math. Bulletin*, **18** (1994), pp. 103–114.

[39] G. Jovančevski, D. Karčicka, S. Božinovski. ILP-model and LP-model of a neural network learning problem, *Math. Bulletin* **22** (1998), pp. 51–60.

[40] S. Božinovski, *Operating Systems and Systems Software I: Von-Neumann Computers and Monoprocessing Operating Systems* (In Macedonian), Zumpres, 1994.

[41] S. Božinovski, *Robotics and Intelligent Manufacturing Systems* (In Macedonian), Gocmar, 1998.

[42] S. Bozinovski, An overlooked identity in Number Theory, *Contributions, Sec. Nat. Math. Biotech. Sci*. **35** (1) (2014), pp.53–56.

[43] S. Bozinovski, A. Bozinovski, Artificial Intelligence and infinity: Infinite series generated by Turing Machines, *Proceedings of IEEE Southeast Con*, Concord, NC, 2017.

ПРОФЕСОРОТ ЧУПОНА И РАЗВОЈОТ НА КОМПЈУТЕРСКИТЕ НАУКИ ВО МАКЕДОНИЈА

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Оваа статија дава преглед на придонесот на професор Чупона во развојот на компјутерските науки во Македонија. Пристапот е од гледна точка на историскиот развој, со распон на години наведени за поедин придонес. Се почнува од 1960-тите години и формирањето на Математичкиот институт со нумерички центар. Потоа се истакнува улогата на Чупона во поттикнувањето на првиот труд напишан на македонски јазик од областа на компјутерските науки. Се наведува неговата улога во пионерските достигнувања на македонската компјутерска наука во светот. Исто така, се наведува неговиот придонес како инспирација за користење на математиката во работата и истражувањето во компјутерските науки.

Клучни зборови: професор Чупона, развој на компјутерските науки, Македонија 1964−2017

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Original scientific paper

ON FREE GROUPOIDS WITH $(xy)^n = x^n y^n$

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We investigate free objects in the variety of groupoids which satisfy the identity $(xy)^n = x^n y^n$. Under certain condition for the groupoid power x^n, i.e. for simple groupoid powers, a canonical description for free groupoids in such varieties is given and they are characterized by the injective groupoids in these varieties.

Key words: variety of groupoids, free groupoid, groupoid powers

INTRODUCTION

In the papers [3,5,6,8,9], Čupona and coauthors investigated free objects in varieties of groupoids satisfying some identities among groupoid powers. Free objects in the variety of groupoids satisfying the law $(xy)^2 = x^2y^2$ are investigated in [4]. Almost 20 years ago, together with Professor Čupona, we obtained a canonical description of free objects in the variety of groupoids satisfying the identity $(xy)^n = x^n y^n$ for some groupoid powers x^n. This result was not published, and the question of finding a canonical description of free objects for an arbitrary groupoid power x^n is still open. In this paper we present a slight improvement of the above mentioned, canonical description.

First, we state some necessary preliminaries.

Let $\boldsymbol{G} = (G,\cdot)$ be a *groupoid*, i.e. an algebra with a binary operation $(x,y) \to xy$ on G. If $a = bc$ for $a,b,c \in G$, we say that b,c are *divisors* of a in $\boldsymbol{G}$. A sequence $a_1, a_2, \ldots$ of elements of G is said to be a *divisor chain* in $\boldsymbol{G}$ if a_{i+1} is a divisor of a_i. We say that $a \in G$ is a *prime* in $\boldsymbol{G}$ if the set of divisors of a in $\boldsymbol{G}$ is empty. A groupoid $\boldsymbol{G} = (G,\cdot)$ is said to be *injective* if $xy = uv$ implies $(x,y) = (u,v)$, for any $x,y,u,v \in G$. By a "free groupoid" we mean "free groupoid in the variety of groupoids" (i.e. an "absolutely free groupoid").

The following characterization of free groupoids is well known (see for example [1], I.1.)

Theorem. 1.1 *A groupoid $\boldsymbol{F} = (F,\cdot)$ is free if and only if (iff) it satisfies the following conditions.*

(1) *Every divisor chain in $\boldsymbol{F}$ is finite.*
(2) *$\boldsymbol{F}$ is injective.*

Then the set B of primes in $\boldsymbol{F}$ is nonempty and it is the unique basis of $\boldsymbol{F}$. ■

Throughout the paper, a free groupoid with basis B will be denoted by $\boldsymbol{F}$ or $\boldsymbol{F}(B)$. For any $v \in F$, we define the *length* $|v|$ and the *set* $P(v)$ *of parts of* v by:

$$|b| = 1, \quad |tu| = |t| + |u|$$
$$P(b) = \{b\},\ P(tu) = \{tu\} \cup P(t) \cup P(u)$$

for every $b \in B,\ t,u \in F$.

GROUPOID POWERS

We recall some definitions, notions and statements from [7].

Let $\boldsymbol{E} = (E,\cdot)$ be a free groupoid with one-element basis $\{e\}$. The elements of E will be denoted by $f, g, h, \ldots$ and called *groupoid powers*.

If $\boldsymbol{G} = (G,\cdot)$ is a groupoid, then each $f \in E$ induces a transformation $f^{\boldsymbol{G}}$ of G (called the *interpretation* of f in $\boldsymbol{G}$) defined by:

$$f^{\boldsymbol{G}}(x) = \varphi_x(f)$$

where $\varphi_x : E \to G$ is the unique homomorphism from $\boldsymbol{E}$ to $\boldsymbol{G}$ such that $\varphi_x(e) = x$. In other words

$$e^{\boldsymbol{G}}(x) = x,\ (fh)^G(x) = f^{\boldsymbol{G}}(x)h^{\boldsymbol{G}}(x),$$

for any $f,h \in E, x \in G$. (For a fixed groupoid $\boldsymbol{G}$ we usually write $f(x)$ instead of $f^{\boldsymbol{G}}(x)$.)

Each $f \in E$ induces a transformation $f^{\boldsymbol{E}}$ of E. We define a new operation " ∘ " on E by:

$$f \circ g = f^{\boldsymbol{E}}(g) = f(g).$$

So, we obtain an algebra $(E,\circ,\cdot)$ with two operations, such that for any $f,g,h \in E$:

$$e \circ f = f \circ e = f$$
$$(fg) \circ h = (f \circ h)(g \circ h).$$

A power $f \in E$ is said to be *irreducible* if $f \neq e$ and $f = g \circ h$ implies $g = e$ or $h = e$.

The following facts for any $f,g,p,q \in E$, $t,u \in F$ can be shown by induction on lengths.

2.1 $|f(t)| = |f||t|$.

2.2 $t \in P(f(t))$.

2.3 $(f(t) = g(u)$ *and* $|t| = |u|)$ *iff* $(f = g$ *and* $t = u)$.

2.4 $(f(t) = g(u)$ *and* $|t| \geq |u|)$ *iff* $(\exists!\, h \in E)(t = h(u)$ *and* $g = h(f))$.

2.5 $(E,\circ,e)$ *is a cancellative monoid.*

2.6 *If the length of a power* f *is a prime integer, then the power* f *is irreducible.*

2.7 *If* $f \circ p = g \circ q$ *and* p,q *are irreducible, then* $f = g$ *and* $p = q$.

2.8 *For* $f \neq e$ *there is a unique sequence of irreducible powers* $p_1, p_2, \dots, p_k$ such that

$$f = p_1 \circ p_2 \circ \dots \circ p_k\ .$$

2.9 *The monoid* $(E,\circ,e)$ *is a free monoid, with a basis the countable set of irreducible powers.*

For a fixed groupoid power $f \in E$ of length *n* we will write x^n instead of $f(x)$. For $n = 2$ there is only one power, $x^2 = x \cdot x$, but for $n \geq 3$ there are different *n*-th powers. For example, $x^3 = x^2 \cdot x$ and $x^3 = x \cdot x^2$ are different powers, and they are the only powers of length three. There are five different powers of length four: $x^4 = x^3 \cdot x$, $x^4 = x^2 \cdot x^2$, $x^4 = x \cdot x^3$, $x^4 = (x \cdot x^2) \cdot x$ and $x^4 = x \cdot (x \cdot x^2)$. It is well known that there are $\frac{(2n-2)!}{n!(n-1)!}$ groupoid powers of length *n*, i.e. *n*-th powers (see, for example [2] page 125, or [7] (1.8)).

A CLASS OF GROUPOIDS DETERMINED BY GROUPOID POWERS

Let $f \in E$ be a groupoid power of length *n* and let *B* be a nonempty set. We will present a specific construction of a groupoid, denoted by $\boldsymbol{R}(f,B)$, determined by f and *B*.

If $\boldsymbol{G} = (G,\cdot)$ is a given groupoid, for any nonnegative integer k we define a transformation $(k): x \to x^{(k)}$ of *G* as the *k*-th power of f in the monoid $(E,\circ,e)$, i.e.

$$x^{(0)} = x,\ x^{(k+1)} = f(x^{(k)}).$$

Using the notion x^n instead of f, we have:

$$x^{(0)} = x,\ x^{(k+1)} = \left(x^{(k)}\right)^n.$$

Since a free groupoid $\boldsymbol{F}$ is injective, it follows that the transformation (k) is injective on $\boldsymbol{F}$, for any $k \geq 0$. Thus, for each $k \geq 0$, there exists an injective partial transformation $(-k): x \to x^{(-k)}$ on $\boldsymbol{F}$ defined by:

$$y^{(-k)} = x \text{ iff } x^{(k)} = y.$$

For any $u \in F$, there exists a largest integer m, such that $u^{(-m)} \in F$. We denote this integer by $[u]$ and call it the *exponent* of u in F.

It is easy to show that the following facts are true for all $u, v \in F$ and all integers t and s.

3.1 $u^{(t)} \in F$ *iff* $t + [u] \geq 0$.

3.2 *If* $t + [u] \geq 0$, *then* $\left|u^{(t)}\right| = n^t|u|$.

3.3 *If* $t + [u] \geq 0$ *and* $t + s + [u] \geq 0$, *then* $\left(u^{(t)}\right)^{(s)} = u^{(t+s)}$.

3.4 *If* $t + [u] \geq 0$ *and* $s - t + [v] \geq 0$, *then* $(u^{(t)} = v^{(s)}$ *iff* $u = v^{(s-t)})$.

Definition 3.1 We define $R(f,B)$, as the least subset of F such that $B \subseteq R(f,B)$ and:

$vw \in R(f,B)$ iff
$[(vw = f(u)$ for some $u \in R(f,B))$ or
$(v,w \in R(f,B)$ and $min\{[v],[w]\} = 0)]$.

We will often write R instead of $R(f,B)$.

Let $S = R \backslash \{u^{(1)} = f(u) | u \in R\}$.

Proposition 3.5 *For every* $h \in E$ *and* $x \in F$,

$$h(x) \in S \text{ implies } x \in R.$$

Proof. The proof is by induction on the length $|h|$ of h. For $|h| = 1$, $h(x) = x \in S$ implies $x \in R$.

Assume that for any $g \in E$ with $|g| < k$, $g(x) \in S$ implies $x \in R$. Let $h = h_1h_2$ and $|h| = k$. Then $h(x) = h_1(x)h_2(x) \in S \subseteq R$ implies that $h_1(x), h_2(x) \in R$ and $min\,\{[h_1(x)],[h_2(x)]\} = 0$, i.e. $[h_i(x)] = 0$ for some $i \in \{1,2\}$. This implies that $h_i(x) \in S$, and the inductive hypothesis, since $|h_i| < k$, implies that $x \in R$. ■

Proposition 3.6 *For every* $u \in F$,

$$u^{(1)} = f(u) \in R \text{ iff } u \in R.$$

Proof. The definition of R implies that, if $u \in R$, then $u^{(1)} = f(u) \in R$.

Let $u^{(1)} \in R$. If $u^{(1)} = v^{(1)}$ for some $v \in R$, then, since the transformation (1) is injective, it follows that $u = v \in R$. If $u^{(1)} \neq v^{(1)}$ for every $v \in R$, i.e. $u^{(1)} \in S$, then Proposition 3.5 implies that $u \in R$. ■

Proposition 3.7 *If for an integer* t *and* $u \in F$, $t + [u] \geq 0$, *then* $(u^{(t)} \in R$ *iff* $u \in R)$.

Proof. The proof is by induction on t, starting from $-[u]$, using the fact 3.3 and Proposition 3.6. ■

Proposition 3.8 *For every* $u, v \in R(f,B)$,

$$\left(u^{(-m)}v^{(-m)}\right)^{(m)} \in R(f,B),$$

where $m = min\{[u],[v]\}$.

Proof. The fact that $m = min\{[u],[v]\}$ implies that $-m+[u] \geq 0$ and $-m+[v] \geq 0$, and so, Proposition 3.7 implies that $u^{(-m)}, v^{(-m)} \in R$. Since $m = [u]$ or $m = [v]$, we have $\left[u^{(-m)}\right] = 0$ or $\left[v^{(-m)}\right] = 0$, and the definition of R implies that

$$\left(u^{(-m)}v^{(-m)}\right)^{(m)} \in R(f,B). \blacksquare$$

If for $u, v \in R(f,B)$ we define $u * v$ by:

$$u * v = \left(u^{(-m)}v^{(-m)}\right)^{(m)},$$

where $m = min\{[u],[v]\}$, then $\boldsymbol{R} = (R(f,B),*)$ is a groupoid.

Proposition 3.9 *For every* $u, v \in R(f,B)$,

$$u^{(1)} * v^{(1)} = (u * v)^{(1)}.$$

Proof. If $m = min\{[u],[v]\}$, then

$$\min\{[u^{(1)}],[v^{(1)}]\} = m+1.$$

The definition of $*$ and the fact 3.3 imply:

$$\begin{aligned} u^{(1)} * v^{(1)} &= \left(\left(u^{(1)}\right)^{(-(m+1))}\left(v^{(1)}\right)^{(-(m+1))}\right)^{(m+1)} \\ &= \left((u)^{(-m)}(v)^{(-m)}\right)^{(m+1)} \\ &= \left(\left((u)^{(-m)}(v)^{(-m)}\right)^{(m)}\right)^{(1)} \\ &= (u * v)^{(1)}. \blacksquare \end{aligned}$$

Let $\mathcal{M}$ be a variety of groupoids. If $\boldsymbol{G} \in \mathcal{M}$, we say that $\boldsymbol{G}$ is an $\mathcal{M}$*-groupoid*, and if it is free in $\mathcal{M}$, we say that it is $\mathcal{M}$*-free.*

For a groupoid power $f \in E$, i.e. x^n, we denote by $\mathcal{M}_f$ the variety of all the groupoids satisfying the identity

$$f(xy) = f(x)f(y), \text{ i.e.}$$
$$(xy)^n = x^n y^n.$$

For the groupoid power $e^2 = ee$, i.e. for the groupoid power x^2, we denote $\mathcal{M}_{e^2}$ by $\mathcal{M}_2$.

We state the following theorems, proven in [4] in their original forms.

Theorem 1. $\boldsymbol{R} = (R(e^2,B),*)$ *is* $\mathcal{M}_2$*-free and the set B is the unique basis for* $\boldsymbol{R}$.

Theorem 2. *An* $\mathcal{M}_2$*-groupoid* $\boldsymbol{H} = (H,\cdot)$ *is* $\mathcal{M}_2$*-free iff the following conditions hold*.

(i) *Every divisor chain in* $\boldsymbol{H}$ *is finite*.

(ii) *If* $x^2 = y^2$, *then* $x = y$.

(iii) *If* $xy = uv$, $x \neq y$ *and* $u \neq v$, *then* $x = u$ *and* $y = v$.

(iv) *If* $x^2 = yz$ *and* $y \neq z$, *then there are* u, v *such that* $x = uv$, $y = u^2$ *and* $z = v^2$.

Then the set P of primes in $\boldsymbol{H}$ *is nonempty and the unique basis for* $\boldsymbol{H}$.

Theorem 3. *If* $\boldsymbol{H}$ *is an* $\mathcal{M}_2$*-free groupoid, then there exist subgroupoids* $\boldsymbol{G}, \boldsymbol{Q}$ *of* $\boldsymbol{H}$, *such that* $\boldsymbol{G}$ *is not* $\mathcal{M}_2$*-free, and* $\boldsymbol{Q}$ *is* $\mathcal{M}_2$*-free with an infinite rank.*

In [4], for any positive integer *n*, the groupoid power e^n, i.e. x^n, is defined as follows:

$$e^1 = e,\ e^{k+1} = e^k e, \text{ i.e.}$$
$$x^1 = e,\ x^{k+1} = x^k x\ .$$

For the groupoid power e^n, we denote $\mathcal{M}_{e^n}$ by $\mathcal{M}_n$.

The generalizations of Theorems 1 – 3, are also discussed in [4]. **Theorem 1'** and **Theorem 3'** are the same as Theorem 1 and Theorem 3, where 2 is replaced by *n*. **Theorem 2'** is obtained from Theorem 2 by replacing 2 by *n* and by replacing (ii), (iii) and (iv) by:

(ii') *If* $x^n = y^n$, *then* $x = y$.

(iii') *If* $xy = uv$, $x \neq y^{n-1}$ *and* $u \neq v^{n-1}$, *then* $x = u$ *and* $y = v$.

(iv') *If* $x^n = yz$ *and* $y \neq z^{n-1}$, *then there are* u, v *such that* $x = uv$, $y = u^n$ *and* $z = v^n$.

We note that Theorems 2 and 2' characterize $\mathcal{M}_2$-free and $\mathcal{M}_n$-free groupoids in the same way as Theorem 1.1 characterizes free groupoids.

It is easy to check that if $uv \in R(e^2,B)$, then $u, v \in R(e^2,B)$, but this is not the case for $R(e^n,B)$ when $n \geq 3$. For example, if $b \in B$ and $n = 3$, then

$$b^{(2)} \in \left(b^{(1)}\right)^{(1)} = \left(b^{(1)}\right)^2 \cdot b^{(1)} \in R(e^3,B),$$

but $\left(b^{(1)}\right)^2 \notin R(e^3,B)$.

From now on, for a groupoid power $g \in E$, of length *p*, we will often write: $g^{\boldsymbol{F}}(x) = x^p$ for $x \in F$, and $g^{\boldsymbol{R}}(x) = x_*^p$ for $x \in R(f,B)$.

The following examples will show that in general, for a groupoid power $f \in E$, $\boldsymbol{R} = (R(f,B),*)$ does not have to belong to $\mathcal{M}_f$, and there are $u \in R$ such that $[u_*^n] = 0$, where $f(x) = x^n$.

Example 3.1. Let $f = e^2 \circ ((e^2)^2 e) \in E$ and let $B = \{a\}$. The length of *f* is 10, and we write $f(x) = x^{10} = ((x^2)^2 x)^2 = (x^5)^2 = x^{(1)}$.

Let $u = a^5 = (a^2)^2 a$. Since $a \in B \subseteq R$, and $[a] = 0$, we have that $a^2 \in R$ and $[a^2] = 0$. This implies that $(a^2)^2 \in R$ and $[(a^2)^2] = 0$. Next, we obtain that $(a^2)^2 a \in R$ and $[(a^2)^2 a] = 0$. All this implies that $u \in R$ and $[u] = 0$.

Now, we calculate $u_*^{10} = ((u_*^2)_*^2 * u)_*^2$, as follows:

$u_*^2 = u^2 = (a^5)^2 = a^{10} = a^{(1)}$;

$(u_*^2)_*^2 = a^{(1)} * a^{(1)} = (a * a)^{(1)} = (a^2)^{(1)}$;

$(u_*^2)_*^2 * u = (a^2)^{(1)} * a^5 = (a^2)^{(1)} a^5$; and

$u_*^{10} = (a^2)^{(1)} a^5 * (a^2)^{(1)} a^5 = ((a^2)^{(1)} a^5)^2$.

We see that $[u_*^{10}] = 0$.

Next, let $v = u$. Then:

$(u * v)_*^{10} = (u_*^2)_*^{10} = \left(a^{(1)}\right)_*^{10} = a^{(2)}$; and

$u_*^{10} * v_*^{10} = (((a^2)^{(1)} a^5)^2)^2$.

Thus, $u_*^{10} * v_*^{10} \neq (u * v)_*^{10}$.

Example 3.2. Let $f = e^3 \circ e^2 \circ e^3 \in E$ and let $B = \{a, b\}$. The length of f is 18, and we write $f(x) = x^{18} = ((x^3)^2)^3 = x^{(1)}$. Let $u = ((a^3)^3)^2$. Since $a \in B \subseteq R$, and $[a] = 0$, we have that $a^2 \in R$ and $[a^2] = 0$. This implies that $a^3 = a^2 a \in R$ and $[a^3] = 0$. Next, $(a^3)^2 \in R$ and $[(a^3)^2] = 0$. This, together with $a^3 \in R$, implies that $((a)^3)^3 \in R$ and $[(a^3)^3] = 0$, and so, $u \in R$ and $[u] = 0$.

Now, we calculate $u_*^{18} = ((u_*^3)_*^2)_*^3$ as follows:

$u_*^2 = u * u = u^2$ and $[u^2] = 0$;

$u_*^3 = u_*^2 * u = u^2 * u = u^3$
$\quad = (((a^3)^3)^2)^3 = (a^3)^{(1)}$;

$(u_*^3)_*^2 = (a^3)^{(1)} * (a^3)^{(1)} = ((a^3)^2)^{(1)}$;

$((u_*^3)_*^2)_*^2 = ((a^3)^2)^{(1)} * ((a^3)^2)^{(1)}$
$\quad = (((a^3)^2)^2)^{(1)}$;

$((u_*^3)_*^2)_*^3 = (((a^3)^2)^2)^{(1)} * ((a^3)^2)^{(1)}$
$\quad = (((a^3)^2)^3)^{(1)} = \left(a^{(1)}\right)^{(1)} = a^{(2)}$.

We see that $[u_*^{18}] = 2$, while $[u] = 0$.

In the same way, for $v = ((b^3)^3)^2$, we obtain that $v_*^{18} = b^{(2)}$.

The previous calculations imply that

$$u_*^{18} * v_*^{18} = a^{(2)} * b^{(2)} = (ab)^{(2)}.$$

In the calculation of $(u * v)_*^{18}$, we have:

$u * v = uv$; $\quad (u * v)_*^3 = (uv)_*^3 = (uv)^3$;

$((u * v)_*^3)_*^2 = ((uv)^3)_*^2 = ((uv)^3)^2$; and

$(u * v)_*^{18} = (((uv)^3)^2)_*^3 = (((uv)^3)^2)^3$
$\quad = (uv)^{(1)}$.

Since $(ab)^{(2)} \neq (uv)^{(1)}$, it follows that

$$u_*^{18} * v_*^{18} \neq (u * v)_*^{18}.$$

We see that the groupoid powers in the previous examples are not irreducible, and moreover, the groupoid power $x^n = (x^p)^q$ has $(x^q)^2$ as its part, i.e. $(x^q)^2 \in P(x^n)$. That is why we consider a special class of groupoid powers, called simple.

We say that a groupoid power x^n is *complex,* if $x^n = ((x^p)^r)^q$ for some $p, q \geq 2$ and $r \geq 1$, and $P(x^n)$ contains $(x)^q (x^r)^q$ or $(x^r)^q (x)^q$. We say that a power x^n is *simple,* if it is not complex.

Irreducible groupoid powers are simple. Since any power x^n, for a prime *n*, is irreducible, it follows that it is simple.

$\mathcal{M}_f$-FREE GROUPOIDS

Let $f = gh \in E\backslash\{e\}$. For a given groupoid $\boldsymbol{G} = (G, \cdot)$ let $T(f, G) \subseteq G \times G$ be defined as:

$$T(f, G) = \{(g(u), h(u)) | u \in G\}.$$

With the notation $f(x) = x^n = x^p x^q$,

$$T(f, G) = \{(u^p, u^q) | u \in G\}.$$

Theorem 4.1 *Let $f = gh$, $g, h \in E\backslash\{e\}$ and with the notation $f(x) = x^n = x^p x^q$, let a groupoid $\boldsymbol{H} = (H, \cdot)$ satisfies the following conditions.*

(i) *Every divisor chain in $\boldsymbol{H}$ is finite.*

(ii) *If $x^n = y^n$ in $\boldsymbol{H}$, then $x = y$.*

(iii) *If $xy = uv$ in $\boldsymbol{H}$, and $xy \neq z^n$ for each $z \in G$, then $x = u$ and $y = v$.*

(iv) *If $x^n = yz$ in $\boldsymbol{H}$ and $(y, z) \notin T(f, H)$, then there are $u, v \in H$, so that $x = uv$, $y = u^n$ and $z = v^n$.*

Then, the groupoid $\boldsymbol{H}$ is $\mathcal{M}_f$-free and the set B of primes in $\boldsymbol{H}$ is nonempty and is the unique basis of $\boldsymbol{H}$.

Proof. The proof is almost the same as the proof of Proposition 2.3 from [4], which is in fact Theorem 4.1 for $f = e^2$, i.e. for the power x^2. The only difference is the following.

The conditions (ii), (iii) and (iv), imply that, for the power x^2, any element $u \in H$ has at most three divisors (shown in [4]), while for any other power, any element $u \in H$ has at most four divisors. The proof of this for a power different than x^2 is as follows. Let $u \in H$.

If u is prime, then it has 0 divisors. If u is not prime, we consider two cases.

Case 1. For any $x \in H$, $u \neq x^n$. Then, the condition (iii) implies that u has at most two divisors.

Case 2. For some $x \in H$, $u = x^n = x^p x^q$. The condition (ii) implies that the element x is unique. If x is prime and $u = yz$, then $(y, z) \notin T(f, H)$ would imply that there are $v, w \in H$, so that $x = vw$, that is not possible. Hence, for x prime, u has at most two divisors. If x is not prime, i.e. if $x = vw$, then $u = x^n = x^p x^q = v^n w^n$, and conditions (ii) and (iv) imply that u has at most four divisors. ■

Theorem 4.2 *If $f(x) = x^n$, and $u_*^n = u^n$ for every $u \in (R(f, B), *)$, then $(R(f, B), *)$ satisfies the conditions* (i) *to* (iv), *from Theorem 4.2, and so it is $\mathcal{M}_f$-free with basis B.*

Proof. Let $x^n = x^p x^q$.

If $x * y = z$, then $|z| > |x|$, $|z| > |y|$, and this implies that $\boldsymbol{R}$ satisfies (i).

If $x_*^n = y_*^n$, then $x^n = y^n$ in $\boldsymbol{F}$, and so $x = y$. Hence, $\boldsymbol{R}$ satisfies (ii).

If $x * y = u * v$ and $x * y \neq z_*^n$ for any $z \in R$, then $\min\{[x],[y]\} = 0 = \min\{[u],[v]\}$. This implies that $x * y = xy$, $u * v = uv$, and $xy = uv$ in $\boldsymbol{F}$. So, $x = u$ and $y = v$. Hence, $\boldsymbol{R}$ satisfies the condition (iii).

Let $x_*^n = y * z$ and $(y,z) \notin T(f,R)$.

If $\min\{[y],[z]\} = 0$, then

$$x^p x^q = x^n = x_*^n = y * z = yz$$

and so, $(y,z) \in T(f,R)$. Hence, $\min\{[y],[z]\} > 0$, and this implies that there are $u, v \in R$, such that $y = u^n = u_*^n$, $z = v^n = v_*^n$, and $x^n = (u * v)^n$, i.e. $x = u * v$. Hence, $\boldsymbol{R}$ satisfies (iv). ■

Theorem 4.3 *Let $f \in E$ be a simple groupoid power, with $f(x) = x^n$. Then, for every $u \in (R(f,B),*)$,*

$$u_*^n = u^n.$$

Proof. By Proposition 3.9 it is enough to consider $x \in R$ with $[x] = 0$. We will show that $x_*^t = x^t$, for any part x^t of x^n.

(1) Since $[x] = 0$, it follows that $x_*^t = x * x = x^2$.

(2) Let $x_*^t = x^t$, for any part x^t of x^n with $t < k$.

(2.1) Let $x^k = x^q x^s$ be a part of x^n with $q < s$.

Then, $x_*^k = x_*^q * x_*^s = x^q * x^s$.

We will show that $\min\{[x^q],[x^s]\} = 0$, which implies that $x_*^k = x^k$. Assume contrary, that, $x^q = u^n$ and $x^s = v^n$ for some $u, v \in R$. Since $[x] = 0$ and $k \leq n$, it follows that $2 \leq q, s < n$. This, implies that, $x = u^m$ and $x = v^p$ for some $m, p \geq 2$, and $u^n = (u^m)^q$, $v^n = (v^p)^s$, and we obtain that

$$z^n = (z^m)^q = (z^p)^s .$$

Since $q < s$, it follows that $z^n = (z^m)^q = (z^p)^s$, $z^m = (z^p)^r$ and $z^s = (z^r)^q$. With all this, we have: $x^n = ((x^p)^r)^q$ and $x^q x^s = x^q (x^r)^q$ is a part of x^n, i.e. the power x^n is not simple. This is a contradiction.

(2.2) The proof that $x_*^k = x^k$, for $x^k = x^s x^q$ with $q < s$ is the same as the proof in (2.1).

(2.3) Let $x^k = x^q x^s$ be a part of x^n with $q = s$, but possibly different powers x^q, x^s, and let $x^q = u^n$ and $x^s = v^n$ for some $u, v \in R$. Similarly as in (2.2), we obtain that, $x = u^m = v^p$, for some $m, p \geq 2$, and $u^n = (u^m)^q$, $v^n = (v^p)^s$. Now, $q = s$ and $sp = n = qm$, imply that $p = m$. This, together with $u^m = v^p$ in $\boldsymbol{F}$ implies that $u = v$ and z^m, z^p are the same powers. Next, $(u^m)^q = (v^p)^s$ in $\boldsymbol{F}$ implies that z^q, z^s are the same powers. All this implies that, $x^n = ((x^p)^1)^q$ and $x^q x^q = x^q (x^1)^q$ is a part of x^n, i.e. x^n is not simple. Hence, $[x^q] = 0$ or $[x^s] = 0$, and $x_*^k = x^k$. ■

The following generalization of Theorem 1 from [4], follows from Theorems 4.2 and 4.3.

Theorem 4.4 *If $f \in E$ is a simple groupoid power, then $(R(f,B),*)$ is $\mathcal{M}_f$-free with basis B, and satisfies the conditions* (i), (ii), (iii) *and* (iv) *from Theorem 4.1.* ■

The next theorem characterizes $\mathcal{M}_f$-free groupoids, for a simple power f, and it is a generalization of Theorem 2 from [4] and Theorem 1.1. Its proof follows from Theorems 4.1, 4.2 and 4.3.

Theorem 4.5 *Let $f \in E$ be a simple groupoid power. A groupoid $\boldsymbol{H} = (H,\cdot)$ is $\mathcal{M}_f$-free if and only if it satisfies the conditions* (i), (ii), (iii) *and* (iv) *from Theorem 4.1. Then, the set B of primes in H is nonempty and is the unique basis of $\boldsymbol{H}$.* ■

REFERENCES

[1] R.H.Bruck, *A survey of binary systems*, Berlin: Springer-Verlag, 1958.

[2] P. M. Cohn, *Universal algebra*, New York, Harper&Row, 1965.

[3] Gj. Čupona, N. Celakoski, Free groupoids with $x^n = x$, *Proc. First Congress of Math. and Inf. of Macedonia, 1996*, SMIM, (1998), pp. 19–25.

[4] Gj. Čupona, N. Celakoski, Free groupoids with $(xy)^2 = x^2y^2$, *Contributions, Sec. Math. Tech. Sci.*, MANU, **XVII**, 1-2, (1996), pp. 5–17.

[5] Gj. Čupona, N. Celakoski, Free groupoids with $xy^2 = xy$, *Math. Bulletin*, **21** (**47**) (1997), pp. 5–16.

[6] Gj. Čupona, N. Celakoski, On groupoids with the identity $x^2y^2 = xy$, *Contributions, Sec. Math. Tech. Sci.*, MANU, **XVIII**, 1-2 (1997), pp. 5–15.

[7] Gj. Čupona, N. Celakoski, S. Ilić, Groupoid powers, *Math. Bulletin*, **25** (**51**) (2001), pp. 5–12.

[8] Gj. Čupona, N. Celakoski, B. Janeva, Free groupoids with axioms of the form $x^{m+1}y = xy$ and/or $xy^{m+1} = xy$, *Novi Sad J. Math.*, **29** 2 (1999), pp. 131–147 .

[9] Gj. Čupona, S. Ilić, Free groupoids with $x^n = x$, II, *Novi Sad J. Math.*, **29** 1 (1999), pp. 147–154.

ЗА СЛОБОДНИ ГРУПОИДИ СО $(xy)^n = x^n y^n$

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Во трудовите [3,5,6,8,9], Чупона со соработниците ги истражува слободните објекти во многуобразија групоиди кои задоволуваат некои идентитети меѓу групоидни степени. Слободни објекти во многуобразието групоиди дефинирано со идентитетот $(xy)^2 = x^2y^2$ се разгледувани во трудот [4]. Пред повеќе од 20 години, заедно со професор Чупона, добивме каноничен опис на слободни објекти во многуобразието групоиди кои го задоволуваат идентитетот $(xy)^n = x^n y^n$ за некои групоидни степени x^n. Овој резултат не беше публикуван, а прашањето за наоѓање каноничен опис на слободни групоиди за произволен групоиден степен x^n е сеуште отворено. Во овој труд е дадено мало подобрување на резултатот од пред 20 години, односно е даден каоничен опис на слободни групоиди во многуобразието групоиди дефинирано со идентитетот $(xy)^n = x^n y^n$, за едноставни групоидни степени x^n. За такви степени, слободните групоиди се карактеризирани со помош на инјективните групоиди од тоа многуобразие.

Клучни зборови: многуобразие групоиди, слободен групоид, групоидни степени

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LACONIC VARIETIES AND THE MEMBERSHIP PROBLEM

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To the memory of Professor Gjorgji Čupona

We introduce laconic varieties and algebras, inspired by a closely related notion in monoids. After providing basic properties of laconic algebras, we define upper distortion functions for laconic subalgebras and apply it to the Membership Problem.

Key words: Membership Problem, distortion function, laconic variety

INTRODUCTION

Graded monoids were introduced in [8] by Margolis, Meakin, and the author as a tool for proving the decidability of certain instances of the Membership Problem in submonoids of groups, which in turn, by earlier results of Ivanov, Margolis, and Meakin [6], implied the decidability of the Word Problem in certain one-relator inverse monoids. Recently, Silva and Zakharov [10] used graded monoids in relation to algorithmic problems in virtually free groups. The notion we are introducing here, laconic algebra, is not exactly a generalization of the notion of a graded monoid to other varieties, but it is closely modeled on it. One advantage of the slightly changed approach is that the new notion is independent of the choice of generating sets (for graded monoids one had to be careful not to include the identity in the generating set), which makes some of the discussion smoother. On the other hand, there are no laconic monoids, so something is lost in this exchange too. The idea behind the approach is very simple – in some algebras, one can tell that some elements cannot be equal just by looking at the lengths of the terms that represent them.

After introducing laconic algebras and varieties, and providing some basic general properties in Section 2, we define upper distortion functions in Section 3, which are related to the corresponding notion in graded monoids, and show how upper distortion can be applied to solve some instances of the Membership Problem. We end with a simple example.

DEFINITION AND BASIC PROPERTIES

We start by defining laconic algebras and varieties, and establishing some of their basic properties.

Definition 2.1 (Laconic algebra/variety). Let $\mathcal{V}$ be a variety. An algebra $\mathbf{A}$ in $\mathcal{V}$ is *laconic* if, for every free algebra $\mathbf{F}$ of finite rank in $\mathcal{V}$, every homomorphism $\phi : \mathbf{F} \to \mathbf{A}$, and every element a in A, the fiber $\phi^{-1}(a)$ is finite.

The variety $\mathcal{V}$ is *laconic* if it contains at least one nonempty laconic algebra.

Recall that, in a variety without constants, the free algebra of rank 0 is the empty algebra, which is, vacuously, laconic. This (and other reasons) is why the definition insists on the existence of a nonempty laconic algebra.

Example 2.1. The variety of semigroups is laconic. Consider the free semigroup of rank 1, namely $\mathbb{N}^+$. Let X be finite, $\phi : X^+ \to \mathbb{N}^+$ a homomorphism from the free semigroup X^+ to $\mathbb{N}^+$, and $a \in \mathbb{N}^+$. No words over X of length larger than a can be mapped to a under ϕ. Since there are only finitely many words over X of length at most a, the fiber $\phi^{-1}(a)$ contains only finitely many words. Thus, $\mathbb{N}^+$ is a laconic semigroup.

This example indicates why we use the term laconic – any element in a laconic semigroup is represented by only a few (finitely many) words.

The variety of monoids is not laconic. There are laconic varieties with constants. For instance, the variety of semigroups with a "central constant" c, defined by the identities $x(yz) = (xy)z$ and $cx = xc$, is laconic.

Example 2.2. Many of the varieties of grupoids studied by Čupona, his collaborators, Celakoski, Dimovski, Markovski, Janeva, Ilić, and their students, are laconic. For instance, the variety of grupoids defined by a single identity of the form $(xy)^n = x^n y^n$, studied in [2, 4], is laconic, and so are the varieties of monoassociative and biassociative grupoids [3, 5].

Example 2.3. In fact, any variety defined by balanced identities is laconic. In such a variety, the free algebra $\mathbf{F}_1$ of rank 1 is laconic, since any element of length at least k in any term algebra maps to an element of (term) length at least k in $\mathbf{F}_1$ (more on term algebras and lengths later).

Example 2.4. The identities defining laconic varieties do not need to be balanced. For instance, the variety of left zeros, defined by the identity $xy = x$, is laconic. In fact, all algebras in this variety are laconic, since all algebras in this variety are free, all maps between them are homomorphisms, and the f.g. free algebras are precisely the finite ones. Related examples of laconic varieties with non-balanced identities are the varieties of k-left-zero semigroups ($k \geq 0$), studied in [9]. For a laconic variety of semigroups, only these two options are available: either it is defined by balanced identities or all of its f.g. algebras are finite. There are laconic varieties of grupoids with non-balanced identities and infinite f.g. free algebras.

Proposition 2.1 (Closure properties)**.** *Let $\mathcal{V}$ be a laconic variety.*

(a) The subclass of laconic algebras in $\mathcal{V}$ is closed under subalgebras.

(b) The subclass of laconic algebras in $\mathcal{V}$ is closed under inverse images.

(c) The subclass of laconic algebras in $\mathcal{V}$ is closed under arbitrary products.

(d) Any product in $\mathcal{V}$ in which at least one factor is laconic is itself laconic.

(e) All free algebras in $\mathcal{V}$ are laconic.

(f) If $\mathbf{A}$ is laconic and there exists a homomorphism $\psi : \mathbf{B} \to \mathbf{A}$, then $\mathbf{B}$ is laconic.

Proof. (a) Let $\mathbf{A}$ be a laconic algebra and $\mathbf{B} \leq \mathbf{A}$. Any homomorphism $\phi : \mathbf{F} \to \mathbf{B}$ from a free algebra $\mathbf{F}$ of finite rank to $\mathbf{B}$ is a restriction (in codomain) of the homomorphism $\phi' : \mathbf{F} \to \mathbf{A}$, where, for all $f \in F$, we have $\phi'(f) = \phi(f)$. Every ϕ-fiber of an element in B is a ϕ'-fiber, and since $\mathbf{A}$ is laconic, any such fiber is finite. Thus $\mathbf{B}$ is laconic.

(b), (c), (d), and (e) are corollaries of (f).

(f) Let $\phi : \mathbf{F} \to \mathbf{B}$ be a homomorphism from the free algebra $\mathbf{F}$ of finite rank to $\mathbf{B}$. Since $\mathbf{A}$ is laconic, all fibers of the homomorphism $\psi\phi : \mathbf{F} \to \mathbf{A}$ are finite. For any element b in B, the ϕ-fiber of b is a subset of the $\psi\phi$-fiber of $\psi(b)$, which is finite. Thus, $\mathbf{B}$ is laconic. $\square$

Corollary 2.2. *A variety $\mathcal{V}$ is laconic if and only it its free algebra of rank 1 is laconic.*

The property of being laconic is local.

Proposition 2.3 (Laconic is local)**.** *An algebra $\mathbf{A}$ in a laconic variety $\mathcal{V}$ is laconic if and only if every finitely generated subalgebra of $\mathbf{A}$ is laconic.*

Proof. For the forward direction, recall that the class of laconic algebras is closed under subalgebras.

For the backward direction, assume that all finitely generated subalgebras of $\mathbf{A}$ are laconic. Let $\phi : \mathbf{F} \to \mathbf{A}$ be a homomorphism from the free algebra $\mathbf{F}$ of finite rank to $\mathbf{A}$. Since $\mathbf{F}$ has finite rank, the subalgebra $\phi(\mathbf{F})$ of $\mathbf{A}$ is finitely generated, which implies that $\phi(\mathbf{F})$ is laconic. Every ϕ-fiber of an element in A is either empty or a fiber of an element in $\phi(\mathbf{F})$. In both cases, the fiber is finite. Thus, $\mathbf{A}$ is laconic. $\square$

Corollary 2.4. *The class of laconic algebras in a laconic variety $\mathcal{V}$ is closed under directed unions.*

The subclass of laconic algebras in a laconic variety is not, in general, closed under homomorphic images. For instance, finite semigroups are not laconic, but they are images of free semigroups, which are laconic. However, the property is preserved under homomorphic images, provided the fibers of the homomorphism are finite.

Proposition 2.5 (Laconic images)**.** *An algebra $\mathbf{A}$ in a laconic variety $\mathcal{V}$ is laconic if and only if it is a homomorphic image, with finite fibers, of a laconic algebra.*

Proof. For the forward direction, observe that the identity map has finite fibers.

For the backward direction, assume that $\mathbf{B}$ is a laconic algebra and $\psi : \mathbf{B} \to \mathbf{A}$ is a surjective homomorphism with finite fibers. Let $\phi : \mathbf{F} \to \mathbf{A}$ be a homomorphism from the free algebra $\mathbf{F}$ of finite rank to $\mathbf{A}$. By the projective property of the free algebra $\mathbf{F}$, there exist a lift $\phi' : \mathbf{F} \to \mathbf{B}$, such that $\phi = \psi\phi'$. The fibers of ψ are finite by assumption, and the fibers of ϕ' are finite, since $\mathbf{B}$ is laconic. Thus, $\mathbf{A}$ is laconic. $\square$

UPPER DISTORTION AND APPLICATION TO THE MEMBERSHIP PROBLEM

In this section we discuss algorithmic issues and, accordingly, limit our attention to finitely generated algebras in laconic varieties of finite type. Parts of the discussion are valid in wider settings, but we will not attempt to indicate such moments.

Let $\mathcal{V}$ be any variety of finite type and X a finite set. A general way to construct the f.g. free algebra $\mathbf{F}(X)$ in $\mathcal{V}$ is by using the set $T(X)$ of terms over X, and the corresponding term algebra $\mathbf{T}(X)$ (see [1]). The elements of $\mathbf{T}(X)$ are classes of terms that are identified by the identities of $\mathcal{V}$. For a term τ in $T(X)$, the element of $\mathbf{T}(X)$ represented by τ is denoted by $\overline{\tau}$. The length of a term τ in $T(X)$, denoted $|\tau|_X$, is the total number occurrences of k-ary operation symbols, for $k \geq 1$ (symbols for constants are not counted). To emphasize the dependence on X, we sometimes call this length the X-length and we say X-term for an element of $T(X)$ (especially when there are other term algebras and bases around). The set of all X-terms of X-length no greater than n is denoted by $T_n(X)$. The length of an element $\overline{\tau}$ in the term algebra $\mathbf{T}(X)$, denoted $|\overline{\tau}|_X$, is the length of the shortest term in the class of τ. The set of all elements in the term algebra $\mathbf{T}(X)$ of length no greater than n is denoted by $\mathbf{T}_n(X)$. Since X and the type are finite, both $T_n(X)$ and $\mathbf{T}_n(X)$ are finite and, for future reference, we note that $\overline{T_n(X)} = \mathbf{T}_n(X)$.

Let $\mathbf{A}$ be a f.g. algebra in the variety $\mathcal{V}$. One of the ways to give a representation of the algebra $\mathbf{A}$ is through a surjective homomorphism $\psi : \mathbf{T}(X) \to \mathbf{A}$ from a term algebra $\mathbf{T}(X)$ over a finite basis X, along with a description of the corresponding congruence θ on $\mathbf{T}(X)$ such that $\mathbf{T}(X)/\theta \cong \psi(\mathbf{T}(X)) = \mathbf{A}$. Concretely, if we are given a finite set R of pairs in $\mathbf{T}(X)$ that generates the congruence θ, we say that the algebra $\mathbf{A}$ is finitely presented by the pair (X, R). The Word Problem for the finite presentation of $\mathbf{A}$ given by (X, R) asks for an algorithm deciding, for any two terms τ_1 and τ_2 in $T(X)$, if $\overline{\tau}_1\theta\overline{\tau}_2$, that is, if $\psi(\overline{\tau}_1) = \psi(\overline{\tau}_2)$. We take a more general view of the Word Problem as follows. The elements of the algebra $\mathbf{A}$ may be represented in any particular way (sets, functions, diagrams, graphs, matrices, or any other convenient construction). Note that defining ϕ amounts to naming a finite generating system for $\mathbf{A}$ (we say system rather than set, since we may choose, on purpose or unknowingly, the same element from A several times in the system). The Word Problem then asks for an algorithm deciding, given any two terms τ_1 and τ_2 in $T(X)$, if $\psi(\overline{\tau}_1) = \psi(\overline{\tau}_2)$. When such an algorithm exists, we say that the Word Problem for $\mathbf{A}$ is decidable.

Let a f.g. subalgebra $\mathbf{B}$ of the f.g. algebra $\mathbf{A}$ be given by a finite set T of terms in $\mathbf{T}(X)$ such that $\psi(T)$ generates $\mathbf{B}$. The Membership Problem for $\mathbf{B}$ in $\mathbf{A}$ asks for an algorithm deciding, given any term τ in $T(X)$, if $\psi(\overline{\tau}) \in B$. When such an algorithm exists, we say that the Membership Problem for $\mathbf{B}$ in $\mathbf{A}$ is decidable. It is known that the decidability of the Word Problem and the Membership Problem do not depend on the choice of the homomorphism ψ (they are properties of the algebras, not of the representations).

Standing assumptions. We make several standing assumptions.

We consider two varieties $\mathcal{W}$ and $\mathcal{V}$ of finite types $\Omega_{\mathcal{W}}$ and $\Omega_{\mathcal{V}}$, respectively, such that $\Omega_{\mathcal{W}} \supseteq \Omega_{\mathcal{V}}$, the set of identities of $\mathcal{W}$ includes those of $\mathcal{V}$, and $\mathcal{V}$ is laconic (a simple example to have in mind: $\mathcal{W}$ is the variety of groups and $\mathcal{V}$ is the variety of semigroups). Let $\mathbf{A}$ be a f.g. $\mathcal{W}$-algebra, X a finite set, $\psi : \mathbf{T}(X) \to \mathbf{A}$ a representation of $\mathbf{A}$, and $T = \{\tau_1, \ldots, \tau_m\}$ a finite set of X-terms. Since $\mathbf{A}$ can also be considered as a $\mathcal{V}$-algebra, we can consider the $\mathcal{V}$-subalgebra of $\mathbf{A}$ given by $\mathbf{B} = \langle \psi(\tau_1), ..., \psi(\tau_m)\rangle_{\mathcal{V}}$. Let $Y = \{y_1, \ldots, y_m\}$, with the obvious bijection to T, and define a representation $\phi : \mathbf{T}(Y) \to \mathbf{B}$ by $\phi(y_i) = \psi(\tau_i)$, for $i = 1, \ldots, m$.

We are interested in the Membership Problem for $\mathbf{B}$ in $\mathbf{A}$, that is, given arbitrary $\tau \in T(X)$, we want to know if $\psi(\tau) \in B$. In general, the terms in $T(X)$ are of type $\Omega_{\mathcal{W}}$ and those in $T(Y)$ are of type $\Omega_{\mathcal{V}}$. Thus, the terms $\tau.\tau_1, \ldots, \tau_m$ may use operation symbols that are not in $\Omega_{\mathcal{V}}$ and we have a slightly extended view of the Membership Problem, which in its standard setting has $\mathcal{W} = \mathcal{V}$.

Definition 3.1 (Upper distortion)**.** Standing assumptions apply. If $\mathbf{B}$ is laconic, the *actual upper distortion function* for $\mathbf{B}$ in $\mathbf{A}$, with respect to ψ and ϕ, is the function $\hat{f} : \mathbb{N} \to \mathbb{N}$ defined by

$$\hat{f}(n) = \max\{ \ |\bar{t}|_Y \ : \ t \in T(Y), \\ \bar{t} \in \phi^{-1}\psi(\mathbf{T}_n(X)) \ \}.$$

An *upper distortion function* for $\mathbf{B}$ in $\mathbf{A}$ is any function $f : \mathbb{N} \to \mathbb{N}$ that bounds the actual distortion function from above.

Let us quickly verify that the definition of the actual upper distortion function $\hat{f}$ makes sense. The set $\mathbf{T}_n(X)$ is finite, which makes $\psi(\mathbf{T}_n(X))$ finite as well. Since $\mathbf{B}$ is laconic the set $\phi^{-1}\psi(\mathbf{T}_n(X))$ is finite, which means that the maximum exists.

For better understanding, let us also parse the meaning of any upper distortion function f. The set $\psi(\mathbf{T}_n(X)) = \psi(\overline{T_n(X)})$ is the finite set of elements in A that can be represented by an X-term of X-length no greater than n. The set $\phi^{-1}\psi(\mathbf{T}_n(X))$ is then the finite set of all elements in the term algebra $\mathbf{T}(Y)$ that represent the elements in $B \cap \psi(\overline{T_n(X)})$. Since none of the elements in $\phi^{-1}\psi(\mathbf{T}_n(X))$ has Y-length greater than $\hat{f}(n) \le f(n)$, we have

$$B \cap \psi(\overline{T_n(X)}) = \phi\phi^{-1}\psi(\mathbf{T}_n(X)) \subseteq \\ \phi(\mathbf{T}_{f(n)}(Y)) = \phi(\overline{T_{f(n)}(Y)}).$$

In other words, every element of $\mathbf{B}$, representable by an X-term of length at most n, must be representable by a Y-term of length at most $f(n)$. We could say the upper distortion gives an upper bound on the "distortion in length" from a representation of the elements in $\mathbf{B}$ by X-terms (external generators, operation symbols in $\Omega_{\mathcal{W}}$) to a representation by Y-terms (internal generators for $\mathbf{B}$, operation symbols from $\Omega_{\mathcal{V}}$). With this understanding the next results is practically a tautology.

Proposition 3.1 (Membership Problem)**.** *Standing assumptions apply. Assume further that the Word Problem for* $\mathbf{A}$ *(as a* $\mathcal{W}$*-algebra) is decidable,* $\mathbf{B}$ *is laconic, and there is a computable (recursive) upper distortion function* f *for* $\mathbf{B}$ *in* $\mathbf{A}$ *with respect to* ψ *and* ϕ*. Then, the Membership Problem for* $\mathbf{B}$ *in* $\mathbf{A}$ *is decidable.*

Proof. We present an algorithm solving the Membership Problem.

Because Y and the type $\Omega_{\mathcal{V}}$ are finite, we may list all Y-terms by length (first all with length 0, then those with length 1, and so on). For every Y-term $t(y_1, \dots, y_m)$ in this list, we have

$$\phi(t(\overline{y}_1, \dots, \overline{y}_m)) = \\ t(\phi(\overline{y}_1), \dots, \phi(\overline{y}_m)) = \\ t(\psi(\overline{\tau}_1), \dots, \psi(\overline{\tau}_m)) = \psi(t(\overline{\tau}_1, \dots, \overline{\tau}_m)),$$

that is, the Y-term $t(y_1, \dots, y_m)$ represents the same element in B as the X-term $t(\tau_1, \dots, \tau_m)$ does. For every term $t(y_1, \dots, y_m)$ in the list of Y-terms ordered by length, consider the corresponding X-term $t(\tau_1, \dots, \tau_m)$. We can, by the decidability of the Word Problem for $\mathbf{A}$, decide if $t(\tau_1, \dots, \tau_m)$ and τ represent the same element of A. If, at any point, the answer is yes, we may stop and declare that $\psi(\overline{\tau})$ is in B. Assume that the X-length of τ is n. Once we check all terms in $T_{f(n)}(Y)$ and if we still do not have a positive answer, we may stop and declare that $\psi(\overline{\tau})$ is not in B. Indeed, if $\psi(\overline{\tau}) \in B$, then

$$\psi(\overline{\tau}) \in B \cap \psi(\overline{T_n(X)}) \subseteq \phi(\overline{T_{f(n)}(Y)}),$$

which means that, once we verify that $\psi(\overline{\tau}) \notin \phi(\overline{T_{f(n)}(Y)})$, we know that $\psi(\overline{\tau}) \notin B$. $\square$

The previous proposition seems difficult to use, since it is not always clear how one can find an upper distortion function. The following proposition says that if one understands a laconic homomorphic image, which is presumably simpler and easier for analysis, one can just lift any upper distortion function found for the image and use it.

Proposition 3.2 (Lifting)**.** *Standing assumptions apply. Let* $\alpha : \mathbf{A} \to \mathbf{A}'$ *be a surjective* $\mathcal{W}$ *homomorphism,* $\alpha_B : \mathbf{B} \to \mathbf{B}'$ *its restriction to a surjective* $\mathcal{V}$*-homomorphism, where* $B' = \alpha(B) = \alpha_B(B)$*. The term algebra* $\mathbf{T}(X)$ *represents the elements of* $\mathbf{A}'$ *through* $\alpha\psi$ *and the term algebra* $\mathbf{T}(Y)$ *represents the elements of* $\mathbf{B}'$ *through* $\alpha_B\phi$*. If* $\mathbf{B}'$ *is laconic, so is* $\mathbf{B}$*, and any upper distortion function* f' *for* $\mathbf{B}'$ *in* $\mathbf{A}'$*, with respect to* $\alpha\psi$ *and* $\alpha_B\phi$*, is an upper distortion function for* $\mathbf{B}$ *in* $\mathbf{A}$*, with respect to* ψ *and* ϕ*.*

Proof. The algebra $\mathbf{B}$ is laconic as an inverse image of the laconic algebra $\mathbf{B}'$. Let $\bar{t}$ be an element of the term algebra $\mathbf{T}(Y)$. We have

$$\begin{aligned} &\bar{t} \in \phi^{-1}\psi(\mathbf{T}_n(X)) \\ \implies\ &\phi(\bar{t}) \in (B \cap \psi(\mathbf{T}_n(X))) \\ \implies\ &\alpha_B\phi(\bar{t}) \in \alpha\psi(\mathbf{T}_n(X)) \\ \implies\ &\bar{t} \in (\alpha_B\phi)^{-1}(\alpha\psi)(\mathbf{T}_n(X)), \end{aligned}$$

which shows that $\phi^{-1}\psi(\mathbf{T}_n(X)) \subseteq (\alpha_B\phi)^{-1}(\alpha\psi)(\mathbf{T}_n(X))$ and, therefore $\hat{f}(n)$, the maximum length of an element in $\phi^{-1}\psi(\mathbf{T}_n(X))$, is smaller than or equal to $\hat{f}'(n)$, the maximum length of an element in $(\alpha_B\phi)^{-1}(\alpha\psi)(\mathbf{T}_n(X))$. Thus, for any upper distortion function f' for $\mathbf{B}'$ in $\mathbf{A}'$, we have $\hat{f} \leq \hat{f}' \leq f'$. □

Our final result provides a way to adapt a given upper distortion function from one representation to another. First a simple observation is in order. Let $\psi : \mathbf{T}(X) \to \mathbf{A}$ and $\psi' : \mathbf{T}(X') \to \mathbf{A}$ be two representations of $\mathbf{A}$. Let M be the smallest number such that, for each letter $x' \in X'$, there exists an X-term $\tau_{x'}$ of length at most M such that $\psi(\overline{\tau_{x'}}) = \psi'(\overline{x'})$ (such an M must exist, since X' is finite). Let K be the largest arity of a symbol in $\Omega_{\mathcal{W}}$. Then, for any X'-term τ' of length at most n, there exists an X-term of length at most $(M(K-1)+1)n + M$ that represents the same element in $\mathbf{A}$ as $\psi'(\tau')$. In other words, there exists a linear function $g_{X',X}$ such that, for all n, we have $\psi'(\mathbf{T}_n(X')) \subseteq \psi(\mathbf{T}_{g_{X',X}(n)}(X))$. Analogous linear function exists for any rewriting from one representation to another (from one finite generating system to another).

Proposition 3.3 (Change of representation)**.** *Standing assumptions apply. Let* $\psi' : \mathbf{T}(X') \to \mathbf{A}$ *and* $\phi' : \mathbf{T}(Y') \to \mathbf{B}$ *be additional representations of* $\mathbf{A}$ *and* $\mathbf{B}$*, respectively, and let* $\mathbf{B}$ *be laconic. If* f *is an upper distortion function for* $\mathbf{B}$ *in* $\mathbf{A}$ *with respect to* ψ *and* ϕ*, then* f'*, defined by* $f'(n) = g_{Y,Y'}(f(g_{X',X}(n)))$*, is an upper distortion function for* $\mathbf{B}$ *in* $\mathbf{A}$ *with respect to* ψ' *and* ϕ'*.*

Proof. For a term t' in $T(Y')$, if $\overline{t'} \in (\phi')^{-1}\psi'(\mathbf{T}_n(X'))$, then $\phi'(\overline{t'}) \in B \cap \psi'(\mathbf{T}_n(X'))$, which implies that

$$\phi'(\overline{t'}) \in B \cap \psi(\mathbf{T}_{g_{X',X}(n)}(X)) \subseteq$$
$$\phi(\mathbf{T}_{f(g_{X',X}(n))}(Y)) \subseteq$$
$$\phi'(\mathbf{T}_{g_{Y,Y'}f(g_{X',X}(n))}(Y')),$$

and this implies that $|t'|_Y \leq g_{Y,Y'}(f(g_{X',X}(n)))$. □

Example 3.1. Let $\mathcal{W}$ be the variety of groups, $\mathcal{V}$ the variety of semigroups, $X = \{x, y, z\}$, $Y = \{y_1, y_2, y_3\}$, $\mathbf{A}$ the group with presentation $\langle x, y, z \mid xy = zy^{-1}z^3x\rangle$, $\mathbf{B}$ the subsemigroup of $\mathbf{A}$ generated by $\{x, xy, y^3z^{-1}\}$, $\psi : \mathbf{T}(X) \to \mathbf{A}$ the obvious group representation of $\mathbf{A}$, and $\phi : \mathbf{T}(Y) \to \mathbf{B}$ the semigroup representations of $\mathbf{B}$ given by $\phi(y_1) = x$, $\phi(y_2) = xy$, $\phi(y_3) = y^3z^{-1}$. We want to solve the Membership Problem for $\mathbf{B}$ in $\mathbf{A}$.

Let $M_x = \left(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $M_y = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $\mathbf{A}' = \langle M_x, M_y\rangle$, the subgroup of $SL_2(\mathbb{Z})$ generated by M_x and M_y, and α the surjective group homomorphism defined by $\alpha(x) = M_x$ and $\alpha(y) = \alpha(z) = M_y$. To verify that α defines a homomorphism we need to check that $M_xM_y = M_y^3M_x$, which does hold. Let $\mathbf{B}' = \alpha(\mathbf{B}) = \langle M_x, M_xM_y, M_y^2\rangle_{\mathcal{V}}$, that is, $\mathbf{B}'$ is the semigroup generated by the matrices $M_x = \left(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $M_xM_y = \left(\begin{smallmatrix} 3 & 3 \\ 0 & 1 \end{smallmatrix}\right)$ and $M_y^2 = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$. Let $X' = \{x, y\}$, $\psi' : \mathbf{T}(X') \to \mathbf{A}'$ be the group representation of $\mathbf{A}'$ given by $\psi'(x') = M_x$, $\psi'(y') = M_y$, $Y' = \{y_1', y_2', y_3'\}$, and $\phi' : \mathbf{T}(Y') \to \mathbf{B}'$ the semigroup representation of $\mathbf{B}'$ given by $\phi'(y_1') = M_x$, $\phi'(y_2') = M_xM_y$, $\phi'(y_3') = M_y^2$.

An easy induction on the length shows that if τ' is an X'-term of length at most n, and $\psi'(\overline{\tau}') = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then $|a| + |b| \leq 3^{n+1}$. On the other hand, by induction on length, if t' is a Y' term of length at least n' and $\phi'(\overline{t}') = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then $|a| + |b| > n'$. Therefore, $(\phi')^{-1}\psi'(\mathbf{T}_n(X')) \subseteq \mathbf{T}_{3^{n+1}}(Y')$. This shows that the fibers of ϕ' are finite. Since the fibers of ϕ' are finite and the free semigroup $\mathbf{T}(Y')$ is laconic, the semigroup $\mathbf{B}'$ is laconic by Proposition 2.5. Moreover, the function $f(n) = 3^{n+1}$ is an upper distortion function for $\mathbf{B}'$ in $\mathbf{A}'$ with respect to ψ' and ϕ'.

By Proposition 3.3 and the decidability of the Word Problem in one-relator groups [7], we can explicitly determine a computable upper distortion function for $\mathbf{B}'$ in $\mathbf{A}'$ with respect to $\alpha\psi$ and $\alpha_B\phi$, which we can lift, by Proposition 3.2, to an upper distortion function for $\mathbf{B}$ in $\mathbf{A}$ with respect to ψ and ϕ. Thus, by Proposition 3.1, the Membership Problem for $\mathbf{B}$ in $\mathbf{A}$ is decidable.

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but let me limit myself to the following vignette. Almost thirty years ago, I gave my first talk at an international conference, in Potsdam, Germany. Right after the talk, an audience member, Professor Kaarli from University of Tartu, approached me and asked "Are you a student of Čupona?" I answered that I was and he simply said "You can always tell the lion by the trace he leaves." Never again in my career have I received a compliment that made me as happy. On behalf of all that still feel and cherish that trace, thank you Professor Čupona.

REFERENCES

[1] S. Burris, H. P. Sankappanavar, *A course in universal algebra*, volume **78** of *Graduate Texts in Mathematics*, Springer-Verlag, New York-Berlin, 1981.

[2] Gj. Čupona, N. Celakoski, Free groupoids with $(xy)^2 = x^2y^2$, *Contributions, Sec. Math. Tech. Sci.,* MASA, **XVII**, 1-2 (1996), pp. 5–17.

[3] Gj. Čupona, N. Celakoski, S. Ilić, On monoassociative groupoids, *Mat. Bilten*, **26** (2002), pp. 5–16.

[4] D. Dimovski, Gj. Čupona, On free groupoids with $(xy)^n = x^ny^n$, To appear in this volume.

[5] S. Ilić, B. Janeva, N. Celakoski, Free biassociative groupoids, *Novi Sad J. Math.*, **35** (1) (2005), pp. 15–23.

[6] S. V. Ivanov, S. W. Margolis, J. C. Meakin, On one-relator inverse monoids and one-relator groups, *J. Pure Appl. Algebra*, **159** (1) (2001), pp. 83–111.

[7] W. Magnus, Das Identitätsproblem für Gruppen mit einer definierenden Relation, *Math. Ann.*, **106** (1) (1932), pp. 295–307.

[8] S. W. Margolis, J. Meakin, Z. Šuniḱ, Distortion functions and the membership problem for submonoids of groups and monoids, in *Geometric methods in group theory*, volume **372** of *Contemp. Math.*, Amer. Math. Soc., Providence, RI, (2005), pp. 109–129.

[9] S. Markovski, On a class of semigroups, *Mat. Bilten*, **28** (2) (1978), pp. 29–36.

[10] P. V. Silva, A. Zakharov, On finitely generated submonoids of virtually free groups, *Groups Complex. Cryptol.*, **10** (2) (2018), pp. 63–82.

ЛАКОНСКИ МНОГУОБРАЗИЈА И ПРОБЛЕМОТ НА ПРИПАДНОСТ

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Во спомен на професор Ѓорѓи Чупона

Ги воведуваме поимите на лаконски алгебри и многуобразија, инспирирани од близок поим кај моноидите. Откако ќе ги дадеме основните особини на лаконските алгебри, дефинираме функции на горна дисторзија на лаконските подалгебри и ги применуваме кон проблемот на припадност.

Клучни зборови: проблем на припадност, горна дисторзија, лаконско многуобразие

Би сакал да ја искористам оваа прилика да ја истакнам клучната улога што ја имаше професорот Чупона во мојот математички развој. Во тек на четири години, бев негов студент и асистент и, по среќна околност, се здобив со драгоцената привилегија да имам постојан пристап до неговата канцеларија, до полиците со книги и, највәжно, до неговите мисли, и сето тоа безмерно го впивав. Многу нешта би можеле да се спомнат, но ќе се ограничам само на следнава вињета. Безмалку пред триесет години, го одржав своето прво предавање на меѓународна конференција, во Потсдам, Германија. Веднаш по предавањето, учесник на конференцијата, професор Каарли од Универзитетот во Тарту, пријде и ме праша „Дали сте студент на Чупона?" Одговорив дека сум, на што тој само рече „Секогаш ќе го препознаете лавот по трагата што ја остава." Никогаш повторно во кариерата не добив комплимент што толку ме израдувал. Во името на сите што сè уште ја чувствуваат и ценат таа трага, благодарам професоре Чупона.

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Review

ON REDUCTIONS FOR PRESENTATIONS OF VECTOR VALUED SEMIGROUPS: OVERVIEW AND OPEN PROBLEMS

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Dedicated to our beloved Professor Gjorgji Čupona

Finding a satisfactory combinatorial description of an (n, m)-semigroup given by its (n, m)-presentation $\langle B; \Delta\rangle$ is a quite difficult and complex problem. The majority of results obtained so far consider some particular cases or they relate to a special class of presentations of (n, m)-semigroups called vector (n, m)-presentations of (n, m)-semigroups. This is because vector (n, m)-presentations of (n, m)-semigroups induce corresponding binary semigroup presentations, and the question of the existence of a good combinatorial description for $\langle B; \Delta\rangle$ is closely related to the question of the existence of a good combinatorial description for the corresponding induced binary semigroup $\langle B; \Lambda\rangle$. An expository overview of the obtained results is given. We classify conditions under which a good combinatorial description for $\langle B; \Lambda\rangle$ implies word problem solvability for $\langle B; \Delta\rangle$. Furthermore, we state a couple of open problems and consider the application of this ideas in varieties of (n, m)-semigroups, giving suggestions for future investigations.

Key words: (n, m)-semigroup, (n, m)-presentation, reduction, word problem

INTRODUCTION

The development of the theory of multivariable groups (called most often n-ary groups or just n-groups) was initiated, we might say, by the paper [22] of E. L. Post. Later, several authors have made generalizations for n-ary semigroups, semigroups of transformations and algebras of multiplace functions (see for example [16, 17, 23, 24, 25]). Motivated partly by some of these papers, Gj. Čupona and B. Trpenovski have introduced in [1, 30] the notion of an (n, m)-semigroup, that is, a set having a multivariable vector valued associative operation. In continuation we present their definition. The set of positive integers will be denoted by $\mathbb{N} = \{1, 2, 3, \ldots\}$ and the set of the first t positive integers will be denoted by $\mathbb{N}_t = \{1, 2, 3, \ldots, t\}$. Moreover, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N}_{t,0} = \mathbb{N}_t \cup \{0\}$. For a given set $Q \neq \emptyset$, and $t \in \mathbb{N}$, let Q^t be the cartesian product of t copies of Q. If $\mathbf{x} = (a_1, a_2, \ldots, a_t) \in Q^t$, then we write $\mathbf{x} = a_1^t$, and moreover we identify $\mathbf{x}$ with the word $a_1 a_2 \ldots a_t$. For such an $\mathbf{x}$ we say that its length $|\mathbf{x}|$ is t. Let Q^+ be the union of all the cartesian products Q^t, $t \in \mathbb{N}$, which is, by the above identification, the free semigroup generated by Q. Let $n, m, k \in \mathbb{N}$, $n = m + k$ be given. A map $f : Q^n \to Q^m$ is called an (n, m)-operation on Q, and (Q, f) is called an (n, m)-groupoid, i.e. a vector valued groupoid. An (n, m)-groupoid (Q, f) is called an (n, m)-semigroup (vector valued semigroup), if the (n, m)-operation is associative, i.e. if $f(\mathbf{x}f(\mathbf{y})\mathbf{z}) = f(\mathbf{u}f(\mathbf{v})\mathbf{w})$, for any $\mathbf{xyz} = \mathbf{uvw} \in Q^{n+k}$, $\mathbf{y}, \mathbf{v} \in Q^n$. An (n, m)-semigroup (Q, f) is called an (n, m)-group if for each $\mathbf{a} \in Q^k$, $\mathbf{b} \in Q^m$ the equations $f(\mathbf{x}\,\mathbf{a}) = \mathbf{b} = f(\mathbf{a}\,\mathbf{y})$ have solutions $\mathbf{x}, \mathbf{y} \in Q^m$. For $m = 1 = k$, the above notions are the usual notions of binary groupoids, semigroups and groups, and for $m = 1, k > 1$ they are the notions of n-groupoids, n-semigroups and n-groups.

From now on and throughout the paper, we assume that $m \geq 2$.

An (n,m)-groupoid (Q,f) can be considered as an algebra with m n-ary operations $f_1, f_2, \dots, f_m : Q^n \to Q$, such that $f(\mathbf{x}) = f_1(\mathbf{x})f_2(\mathbf{x})\dots f_m(\mathbf{x})$. These operations $f_1, f_2, \dots, f_m$ are called component operations for the (n,m)-operation f. In general, for an associative (n,m)-operation f, the m n-ary operations obtained from f, do not have to be associative. Hence, there is a big difference between studying (n,m)-semigroups and n-semigroups. Vector valued algebraic structures are a generalization of $(2,1)$ structures, and thus they are similar to the binary structures on one hand, but on the other hand they incorporate new ideas and specific properties.

The theory of vector valued structures defined as above, has started to developed in the 80's of the last century. Leaded by one of its founders, Gj. Čupona, a group of macedonian algebraists in just a couple of years gave significant results within this topic. The majority of them are fully cited in the expository papers [2, 4] (though some of them will be referenced separately throughout the paper). Since then, various investigations have been published and upgrades have been made. (See for example [3, 6, 8, 9, 10, 11, 12, 13, 14, 26, 27, 28, 29]). However, some of the ideas given at the very beginning are still looking to be seriously considered and deeply investigated. We do not intend (nor can) state all the results on vector valued semigroups that have been given so far (worldwide). Our aim is to expose the development of the combinatorial theory of (n,m)-semigroups emphasizing the major results on word problem solvability for various classes of (n,m)-semigroups given by their (n,m)-presentations.

The word problem for free vector valued semigroups and groups is solved in [7, 9, 10, 11] by using good combinatorial descriptions for free vector valued semigroups and groups. There have also been made a link between a special class of presentations of (n,m)-semigroups, called vector presentations of (n,m)-semigroup and binary semigroups. They are induced one by another, which will be exposed later in this paper. The majority of the results in continuation concern good combinatorial descriptions that have been obtained for some classes of vector presentations of (n,m)-semigroups. Exploring the above, we have also obtained interesting results for varieties of (n,m)-semigroups. A couple of open problems and suggestions for future investigations will be given at the end.

PRELIMINARIES. PRESENTATIONS OF (n,m)-SEMIGROUPS

The following basic notions were originally given in [2] and [4].

For a given set Q, let

$Q^{m,k} = \{\mathbf{x} | \mathbf{x} \in Q^+, |\mathbf{x}| = m + sk, s \in \mathbb{N}\}$.

If (Q,f) is an (n,m)-semigroup, because of the associative law, the operation f can be extended to an operation, denoted by the same letter, $f : Q^{m,k} \to Q^m$, such that for each $\mathbf{xyz} \in Q^{m,k}$ and $\mathbf{y} \in Q^{m,k}$, $f(\mathbf{x}f(\mathbf{y})\mathbf{z}) = f(\mathbf{xyz})$. As mentioned above, an (n,m)-groupoid (Q,f) can be considered as an algebra with m n-ary operations, $f_j : Q^n \to Q$, $j \in \mathbb{N}_m$. These operations can be extended to an infinite family of operations $f_{j,s} : Q^{m+sk} \to Q$ for $s \in \mathbb{N}$, where for a given s, there are more than one operation $f_{j,s}$. When (Q,f) is an (n,m)-semigroup, for each $s \in \mathbb{N}$, there is only one operation $f_{j,s} : Q^{m+sk} \to Q$ whose union is a map $f_j : Q^{m,k} \to Q$. This leads to the following slightly more general notions. A map $g : Q^{m,k} \to Q^m$ is called a poly-(n,m)-operation and the structure $\boldsymbol{Q} = (Q,g)$ is called a poly-(n,m)-groupoid. A poly-(n,m)-groupoid $\boldsymbol{Q} = (Q,g)$ is called a poly-(n,m)-semigroup if for each $\mathbf{xyz} \in Q^{m,k}$ and $\mathbf{y} \in Q^{m,k}$, $g(\mathbf{x}g(\mathbf{y})\mathbf{z}) = g(\mathbf{xyz})$. There is no essential difference between studying (n,m)-semigroups or poly-(n,m)-semigroups because of the General Associative Law (see [2]). Similarly as above, a poly-(n,m)-groupoid (Q,g) can be considered as an algebra with m poly-n-ary operations, $g_1, g_2, \dots g_m : Q^{m,k} \to Q$ where $g(\mathbf{x}) = g_1(\mathbf{x})g_2(\mathbf{x})\dots g_m(\mathbf{x})$. It is easy to see that the usual notions of universal algebra (i.e. free algebras, varieties) can be extended to (n,m)-semigroups.

Let B be a nonempty set and let $\boldsymbol{B}^+$ be the free semigroup with base B. Let $\Lambda \subseteq B^+ \times B^+$. The pair $\langle B; \Lambda \rangle$ is a presentation of the semigroup $\boldsymbol{B}^+/\overline{\Lambda}$ where $\overline{\Lambda}$ is the smallest congruence on $\boldsymbol{B}^+$ containing Λ. We use the notation $\langle B; \Lambda \rangle = \boldsymbol{B}^+/\overline{\Lambda}$. A reduction for $\langle B;\Lambda\rangle$ is a map assigning a chosen element of a congruence class in $\boldsymbol{B^+}/\overline{\Lambda}$ to every element of the congruence class. In order to extend this to (n,m)-semigroups there have been defined a poly-(n,m)-groupoid, $\boldsymbol{F(B)} = (F(B), f)$, with a base B which is an analogy to the free semigroup B^+ above. Its existence follows from the fact that it is an algebra of type $\Omega = \{\omega_r^j | j \in \mathbb{N}_m, r \in \mathbb{N}\}$. We recall its canonical form. (For more details see [7, 14, 34]).

$B_0 = B$,
$B_{p+1} = B_p \cup (\mathbb{N}_m \times B_p^{m,k})$,
$F(B) = \bigcup_{p\geq 0} B_p$.

By choosing different letters, if necessary, for the elements of B, we will have that no element of B is of the form $(j,\mathbf{x})$. The poly-(n,m)-operation f on $F(B)$ is defined by $f(\mathbf{x}) = (1,\mathbf{x})(2,\mathbf{x})\dots(m,\mathbf{x})$. Hierarchy of the elements of $F(B)$ is a map $\chi : F(B) \to \mathbb{N}_0$ defined by $\chi(u) = \min\{p \,|\, u \in B_p\}$, $u \in F(B)$. Clearly, $\chi(u) = p \Leftrightarrow u \in B_p \backslash B_{p-1}$.

The norm on $F(B)$ is a map $\|\ \| : F(B) \to \mathbb{N}$ defined by induction on χ:

$\|u\| = 1$ for $u \in B_0$,
$\|(i, u_1^{m+sk})\| = \|u_1\| + \ldots + \|u_{m+sk}\|$
for $(i, u_1^{m+sk}) \in B_{p+1}\backslash B_p$.

Thus, the norm $\|(i, u_1^{m+sk})\|$ is the number of appearances of elements from B in (i, u_1^{m+sk}).

For $\mathbf{x} \in F(B)^+$ and $\mathbf{x} = x_1^r$, we define the norm as $\|\mathbf{x}\| = \|x_1\| + \ldots + \|x_r\|$.

We note that the elements of $F(B)$ can be also treated as special words over the alphabet $A = B \cup \mathbb{N}_m \cup \{(\} \cup \{,\} \cup \{)\}$. Hence, every $u \in F(B)$ can be considered as an element of A^+ as well.

For a set $\Delta \subseteq F(B) \times F(B)$, we say that Δ is a set of (n,m)-defining relations on B and the pair $\langle B;\Delta\rangle$ is an (n,m)-presentation of an (n,m)-semigroup. We also say that $\langle B;\Delta\rangle$ is an (n,m)-semigroup presentation. The (n,m)-semigroup whose presentation is $\langle B;\Delta\rangle$ is the factor (n,m)-semigroup $\boldsymbol{F(B)}/\overline{\Delta}$ where $\overline{\Delta}$ is the smallest congruence on $\boldsymbol{F(B)}$ such that $\Delta \subseteq \overline{\Delta}$ and $\boldsymbol{F(B)}/\overline{\Delta}$ is an (n,m)-semigroup. We use the notation $\langle B;\Delta\rangle = \boldsymbol{F(B)}/\overline{\Delta}$. The explicit description of $\overline{\Delta}$ and its properties are given in [4]. Given an (n,m)-presentation $\langle B;\Delta\rangle$ of an (n,m)-semigroup, we are interested in the structure of this (n,m)-semigroup. Analogous to the binary case, a reduction for an (n,m)-presentation $\langle B;\Delta\rangle$ is a map assigning a chosen element of a congruence class in $\boldsymbol{F(B)}/\overline{\Delta}$ to every element of the congruence class.

Proposition 2.1 [4] *A map $\psi : F(B) \to F(B)$ is a reduction for the (n,m)-presentation $\langle B;\Delta\rangle$ if and only if the following properties are satisfied*

(i) $(u,v) \in \Delta \Rightarrow \psi(u) = \psi(v)$

(ii) $\psi(i, \mathbf{x}'(1,\mathbf{y})(2,\mathbf{y})\ldots(m,\mathbf{y})\mathbf{x}'') = \psi(i, \mathbf{x}'\mathbf{y}\mathbf{x}'')$

(iii) $\psi(i, \mathbf{x}'w\mathbf{x}'') = \psi(i, \mathbf{x}'\psi(w)\mathbf{x}'')$

(iv) $u\ \overline{\Delta}\ \psi(u)$

for all $u, v, w, (i, \mathbf{x}'w\mathbf{x}'')$, $(i, \mathbf{x}'(1,\mathbf{y})(2,\mathbf{y})\ldots(m,\mathbf{y})\mathbf{x}'') \in F(B)$. □

We say that $\psi(u)$ is the reduced represent (reduct) for u.

The axiom of choice implies that for any (n,m)-presentation $\langle B;\Delta\rangle$ there exist a reduction. If ψ is a reduction for $\langle B;\Delta\rangle$ such that for any $u \in F(B)$ the reduced represent $\psi(u)$ can be calculated in finitely many steps, ψ is said to be a good (effective) reduction for $\langle B;\Delta\rangle$ and it provides a good combinatorial description for the corresponding (n,m)-semigroup (presented by) $\langle B;\Delta\rangle$.

Proposition 2.2 [4] *A reduction $\psi : F(B) \to F(B)$ for an (n,m)-semigroup presentation $\langle B;\Delta\rangle$ is a homomorphism from $\boldsymbol{F(B)}$ to $(\psi(F(B)); g)$ where*

$\psi(F(B)) = \{u \in F(B) \,|\, \psi(u) = u\}$ *and*
$g(u_1^{m+sk}) = v_1^m \Leftrightarrow v_i = \psi(i, u_1^{m+sk})$, $i \in \mathbb{N}_m$.

Moreover, $\ker\psi = \overline{\Delta}$ *and* $\langle B;\Delta\rangle = (\psi(F(B)), g)$. □

When $\Delta = \emptyset$, then $\langle B;\emptyset\rangle$ is the presentation of the free (n,m)-semigroup generated by B. In 1986, D. Dimovski constructed a canonical form of a free (n,m)-semigroup $S(B)$ generated by B. This was a starting point towards development of a combinatorial theory of vector valued semigroups. Due to its importance, we recall here its construction. (For more details see [7, 14]). We define a map $\psi_0 : F(B) \to F(B)$, by induction on the norm as follows:

(a) $\psi_0(b) = b$, $b \in B$;

(b) Let $u = (i, u_1^{m+sk}) \in F(B)$ and assume that $\psi_0(v) \in F(B)$ is already defined and $\psi_0(v) \neq v$ implies $\|\psi_0(v)\| < \|v\|$ for all $v \in F(B)$, with $\|v\| < \|u\|$. Then

$v_\lambda = \psi_0(u_\lambda)$ is well defined for all $\lambda \in \mathbb{N}_{m+sk}$ and thus $v = (i, v_1^{m+sk}) \in F(B)$.

(b1) If there exists a $\lambda' \in \mathbb{N}_{m+sk}$ such that $v_{\lambda'} \neq u_{\lambda'}$ then $\|v\| < \|u\|$ and

$$\psi_0(u) = \psi_0(v).$$

(b2) If $v_\lambda = u_\lambda$ for all $\lambda \in \mathbb{N}_{m+sk}$ and if $u = (i, u_1^j(1,\mathbf{x})...(m,\mathbf{x})u_{j+m+1}^{m+sk})$ where $\mathbf{x} \in F(B)^{m,k}$ and j is the smallest such index, then

$$\psi_0(u) = \psi_0(i, u_1^j \mathbf{x} u_{j+m+1}^{m+sk}).$$

(b3) If u satisfies neither (b1) nor (b2), then $\psi_0(u) = u$.

Proposition 2. 3 [7, 14] *The map ψ_0 is a good reduction for $\langle B; \emptyset\rangle$.* □

Proposition 2.2 together with Proposition 2.3 imply that $(\psi_0(F(B)), g) = (S(B), g)$ is a free (n, m)-semigroup generated by B. By induction on the norm, we say that an element $u = (i, u_1^{m+sk}) \in F(B)$ is reducible if u_j is reducible for some j, or if $u = (i, u_1^j(1,\mathbf{x})\dots(m,\mathbf{x})u_{j+m+1}^{m+sk})$. Otherwise we say that u is irreducible. With this notion, $S(B)$ is the set of all the irreducible elements in $F(B)$.

The construction above opened new investigation possibilities: To obtain good combinatorial descriptions for various $\langle B; \Delta\rangle$ and to explore the circumstances under which it might be possible. The common approach is to manage to construct a good reduction for $\langle B; \Delta\rangle$ (if possible), a task that is usually quite complicated to achieve. A couple of results on good combinatorial descriptions for some particular $\langle B; \Delta\rangle$ can be found in [4]. In [32] there have been defined a sequence of (n, m)-semigroup presentations $\langle B; \Delta_p\rangle$, for which good reductions have been constructed and consequently, good combinatorial description for such (n, m)-semigroups have been obtained. In [33] we have constructed good reductions for a class of (n, m)-presentations of (n, m)-semigroups that incorporate binary relations within the corresponding (n, m)-relations Δ, under certain conditions. Namely, given a semigroup presentation $\langle B; \Lambda\rangle$ with a good reduction φ that satisfies a pair of conditions, we have defined an associated (n, m)-semigroup presentation $\langle B; \Delta\rangle$ and derived a good reduction ψ for $\langle B; \Delta\rangle$. As a consequence, good combinatorial description of the corresponding (n, m)-semigroup has been given. All this led to a conclusion that valuable results might be obtained by linking (n, m)-semigroup presentations and binary semigroups presentations. In [4], the authors have defined a special class of (n, m)-semigroup presentations, closely related to binary semigroups presentations, called vector (n, m)-presentations of (n, m)-semigroups, and thus made this idea possible. A set of vector (n, m)-defining relations induces also a set of binary relations, i.e. a presentation of a binary semigroup. Under certain conditions for this binary semigroup presentation, there have been obtained good combinatorial descriptions for various classes of (n, m)-semigroups given by their vector (n, m)-presentations. For some vector (n, m)-presentations, the obtained good combinatorial descriptions imply word problem solvability. The aim of this paper is to give an overview of these results.

VECTOR PRESENTATIONS OF (n, m)-SEMIGROUPS. REDUCTIONS

The following definition was originally given in [4] and improved in [34].

Definition 3.1 [34] For an (n, m)-presentation $\langle B; \Delta\rangle$ of an (n, m)-semigroup, we say that it is a vector (n, m)-presentation of an (n, m)-semigroup, in short vector (n, m)-presentation, and that Δ is a set of vector (n, m)-relations, if the following conditions are satisfied:

(1) if $(i, \mathbf{x}) \;\Delta\; (j, \mathbf{y})$, then $i = j$ and $\mathbf{x}, \mathbf{y} \in B^{m,k}$;

(2) if $(i, \mathbf{x}) \;\Delta\; (i, \mathbf{y})$, then $(j, \mathbf{x}) \;\Delta\; (j, \mathbf{y})$ for every $j \in \mathbb{N}_m$;

(3) if $(i, \mathbf{x}) \;\Delta\; b$ for some $b \in B$, then $\mathbf{x} \in B^{m,k}$ and there is $b_1^m \in B^m$, such that $b_i = b$ and for each $j \in \mathbb{N}_m$, $(j, \mathbf{x}) \;\Delta\; b_j$;

(4) if $b \;\Delta\; (i, \mathbf{x})$ for some $b \in B$, then $\mathbf{x} \in B^{m,k}$ and there is $b_1^m \in B^m$, such that $b_i = b$ and for each $j \in \mathbb{N}_m$, $b_j \;\Delta\; (j, \mathbf{x})$; and

(5) $\Delta \cap B \times B = \emptyset$.

In other words, an (n, m)-presentation is a vector (n, m)-presentation if only the (n, m)-operation is used in the defining relations.

Example 3.1. [34] Let $B = \{a, b\}$ and let Δ be the following set:

$\Delta = \{((1, aabb), a), ((2, aabb), b),$
$((1, aaabbb), b), ((2, aaabbb), a),$
$((1, aaa), a), ((2, aaa), a)\}$

The relation from Δ can be written in the form: $[aabb] = ab$, $[aaabbb] = ba$ and $[aaa] =$

aa, and they imply that: $ba = [aaabbb] = [a[aabb]b] = [aabb] = ab$. Thus, in the $(3,2)$-semigroup with the given presentation, a and b have to be identified. The relation $[aaa] = aa$ implies that the $(3,2)$-semigroup whose presentation is $\langle B;\Delta\rangle$ is $(A,[\ ])$ where $A = \{a\}$ and $[aaa] = aa$. Note that we did not use the component operations of the $(3,2)$-operation $[\ \ ]$ for the defining relations. Various examples of vector (n,m)-presentations can be found in [4, 31, 34].

Definition 3.2 [34] For a vector (n,m)-presentation $\langle B;\Delta\rangle$ we define a binary semigroup presentation $\langle B;\Lambda\rangle$, where $\Lambda = \Lambda' \cup \Lambda'' \cup \Lambda'''$, $\Lambda' \subseteq B^{m,k}\times B^{m,k}$, $\Lambda'' \subseteq B^{m,k}\times B^{m}$, $\Lambda''' \subseteq B^{m}\times B^{m,k}$ and $\Lambda',\Lambda'',\Lambda'''$ are defined by:

(1) $\mathbf{x}\ \Lambda'\ \mathbf{y}$ if and only if $(1,\mathbf{x})\ \Delta\ (1,\mathbf{y})$;

(2) $\mathbf{x}\ \Lambda''\ b_1^m$ if and only if $(j,\mathbf{x})\ \Delta\ b_j$ for each $j\in\mathbb{N}_m$;

(3) $b_1^m\ \Lambda'''\ \mathbf{x}$ if and only if $b_j\ \Delta\ (j,\mathbf{x})$ for each $j\in\mathbb{N}_m$.

We say that Λ is induced by Δ, and $\langle B;\Lambda\rangle$ is induced by $\langle B;\Delta\rangle$.

The binary semigroup presentation induced by the vector $(3,2)$-presentation in Example 3.1 is $\langle a,b\,; aabb = ab, aaabbb = ba, aaa = aa\rangle$. Note that the semigroup with this presentation is not trivial, i.e. has more than one element, while the $(3,2)$-semigroup with the corresponding vector $(3,2)$-presentation in Example 3.1 is trivial, i.e. has only one element.

Proposition 3.1 [34] *The class of vector (n,m)-presentations is equivalent to the class of semigroup presentations $\langle B;\Lambda\rangle$ where $\Lambda \subseteq B^{m,k}\times B^{m,k}\cup B^{m,k}\times B^{m}\cup B^{m}\times B^{m,k}$ i.e. there is a bijection between these two classes.*

We note that the empty set is a set of vector (n,m)-relations, that induces a presentation of a free binary semigroup in which the word problem is solvable. But by no means this implies directly that the word problem for free vector valued semigroups is solvable. Establishing the $1-1$ correspondence above, it seemed more achievable to focus on constructing good reductions for vector (n,m)-presentations $\langle B;\Delta\rangle$ and to explore the circumstances under which it might be possible. A couple of investigations have been made in [31]. Here we give some of the conclusions.

Let $\langle B;\Delta\rangle$ be a vector (n,m)-presentation of an (n,m)-semigroup. Providing that there exists a good reduction φ for its induced binary presentation $\langle B;\Lambda\rangle$, a good reduction ψ for $\langle B;\Delta\rangle$ has been constructed in the following cases:

i) If none of the pairs in Λ has length m i.e. if $\Lambda\subseteq B^{m,k}\times B^{m,k}$;

ii) If $\varphi(b_1^m) = b_1^m$ for all $b_1^m\in B^m$;

iii) If φ reduces the length on B^+.

It is easy to notice that i) and iii) are special cases of ii), however we give them independently, since i) was the first conclusion we have obtained and then realized that analogical construction works for wider classes satisfying ii). These results have their improved versions and will be stated as theorems in the next section. Regarding the condition iii), we have proved that the existence of a reduction φ for $\langle B;\Lambda\rangle$ that reduces the length on B^+, allows a construction of a reduction ψ for $\langle B;\Delta\rangle$ that will reduce the norm on $F(B)$.

Theorem 3.2 [31] *Let $\langle B;\Delta\rangle$ be a vector (n,m)-presentation of an (n,m)-semigroup and let φ be a reduction for its induced binary presentation $\langle B;\Lambda\rangle$ satisfying*

$\varphi(x)\neq x \Longrightarrow |\varphi(x)| < x|,\ x\in B^+$.

Then there exists a good reduction ψ for $\langle B;\Delta\rangle$. □

Further improvements of these results and also new once have been obtained thanks to the suggestion of one of the anonymous referees of the paper [34]: to switch the investigations to the language of abstract rewriting systems, instead in the language of reductions only. Straightforward and clearer proofs have been provided through confluent rewriting systems, instead of using good reductions only. Moreover, additional results and important conclusions have been obtained.

ABSTRACT REWRITING SYSTEMS - IMPROVED RESULTS

An abstract rewriting system (ARS) is the most general notion about specifying a set of objects and rules that can be applied to transform them. An ARS is a set A, whose elements are usually called objects, together with a binary relation on A, traditionally denoted by $\longrightarrow$, and called the reduction relation or rewrite relation (rule). Detailed definitions and properties of (abstract) rewriting systems can be found, for example, in [18, 19]. We note that, if a reduction is obtained from a confluent, terminating abstract rewriting system (ARS), then it is a good reduction, and moreover the reduced represent for u is the unique normal form (UNF) for u. (For more details see [18, 19, 34]).

In the construction of a canonical form of a free (n,m)-semigroup (originally given in [7] and introduced above), the good reduction ψ_0 for $\langle B;\emptyset\rangle$ was explicitly defined by induction on the norm. In terms of ARS, the reduction rule corresponding to the map ψ_0 is given by:

$$(j, \mathbf{x}(1,\mathbf{y})(2,\mathbf{y})\dots(m,\mathbf{y})\mathbf{z}) \longrightarrow (j, \mathbf{xyz})$$

With this reduction (or rewrite) rule for $F(B)$, the proof in [7], adjusted to the language of ARS, shows that the ARS obtained by the above rule is confluent and terminating. Hence, it is canonical (also called "complete" or "uniquely terminating"), and the UNF for any $u \in F(B)$ is $\psi_0(u)$. Here we give the results obtained in [34], by the language of ARS.

Theorem 4.1 [34, 31] *Let $\langle B;\Delta\rangle$ be a vector (n,m)-presentation and let $\langle B;\Lambda\rangle$ be its induced semigroup presentation.*

(a) If there is a reduction φ for $\langle B;\Lambda\rangle$ satisfying the condition

(4.1) $\varphi(b_1^m) = b_1^m$ *for all* $b_1^m \in B^m$,

then there is a reduction ψ for $\langle B;\Delta\rangle$ and $(\psi(F(B)), g)$ where

$$g(u_1^n) = \psi(1,u_1^n)\psi(2,u_1^n)\dots\psi(m,u_1^n),$$

is the (n,m)-semigroup $\boldsymbol{F(B)}/\overline{\Delta}$. Moreover, if the reduction φ for $\langle B;\Lambda\rangle$ is good, then the reduction ψ for $\langle B;\Delta\rangle$ is good and the word problem for $\langle B;\Delta\rangle$ is solvable.

(b) If there is a canonical ARS $\mathbf{C}$ for $\langle B;\Lambda\rangle$, such that the UNF of b_1^m is b_1^m for every $b_1^m \in B^m$, then there is a canonical ARS $\mathbf{D}$ for $\langle B;\Delta\rangle$, and the word problem for $\langle B;\Delta\rangle$ is solvable. □

Remark. The canonical ARS $\mathbf{D}$ for $\langle B;\Delta\rangle$ in the proof of Theorem 4.1 is obtained by the rewriting rules of $\mathbf{C}$ and the rewriting rules of an ARS $\mathbf{A}$ we have constructed, that is terminating and (locally) confluent (Church Rosser), and thus canonical (by Newman Lemma). But all this modulo the UNF for $\mathbf{x}$ obtained by $\mathbf{C}$, since in the rewriting rules, UNF for $\mathbf{x}$ is a black box. (For more details see [34, 18, 21]).

The next result, although a corollary of Theorem 4.1 is stated as a Theorem, because: it can be proven independently of Theorem 4.1, by using a simpler ARS; it was the first step toward the proof of Theorem 4.1; and it is easier to check if it can be applied to a given vector (n,m)-presentation.

Theorem 4.2 [34, 31] *Let $\langle B;\Delta\rangle$ be a vector (n,m)-presentation and $\langle B;\Lambda\rangle$ its induced semigroup presentation. If $\Lambda \subseteq B^{m,k} \times B^{m,k}$, then any reduction φ for $\langle B;\Lambda\rangle$, generates a reduction ψ for the vector (n,m)-presentation $\langle B;\Delta\rangle$ and $(\psi(F(B)), g)$ where*

$$g(u_1^n) = \psi(1,u_1^n)\psi(2,u_1^n)\dots\psi(m,u_1^n),$$

is the (n,m)-semigroup $\boldsymbol{F(B)}/\overline{\Delta}$.

Moreover:

a) if there is a good reduction φ for $\langle B;\Lambda\rangle$, then there is a good reduction ψ for $\langle B;\Delta\rangle$ and the word problem for $\langle B;\Delta\rangle$ is solvable;

b) if there is a canonical ARS for $\langle B;\Lambda\rangle$, then there is a canonical ARS for $\langle B;\Delta\rangle$ and the word problem for $\langle B;\Delta\rangle$ is solvable. □

The following theorem is an improvement of Theorem 4.1.

Theorem 4.3 [34] *Let $\langle B;\Delta\rangle$ be a vector (n,m)-presentation and let φ be a reduction for the induced semigroup presentation $\langle B;\Lambda\rangle$ such that its restriction to B^m is injective. Then, the reduction φ for $\langle B;\Lambda\rangle$, generates a reduction ψ for the vector (n,m)-presentation $\langle B;\Delta\rangle$ and $(\psi(F(B)), g)$ where*

$$g(u_1^n) = \psi(1,u_1^n)\psi(2,u_1^n)\dots\psi(m,u_1^n),$$

is the (n,m)-semigroup $\boldsymbol{F(B)}/\overline{\Delta}$.

Moreover, if B is finite and the reduction φ for $\langle B;\Lambda\rangle$ is good, then the reduction ψ for $\langle B;\Delta\rangle$ is good and the word problem for $\langle B;\Delta\rangle$ is solvable. □

In the following result we have improved Theorem 4.1 - for finite generating sets, obtaining that when B is finite, no extra conditions on the good reduction φ are required.

Theorem 4.4 [34] *Let $\langle B;\Delta\rangle$ be a vector (n,m)-presentation and let B be a finite set. Then, the existence of a good reduction for the induced semigroup presentation $\langle B;\Lambda\rangle$ implies existence of a good reduction for the vector (n,m)-presentation $\langle B;\Delta\rangle$, and the word problem for $\langle B;\Delta\rangle$ is solvable.* □

OPEN PROBLEMS

We still do not have answers to the following questions.

1) Does the existence of good reduction for the induced semigroup presentation $\langle B;\Lambda\rangle$ whose restriction to B^m is injective imply existence of a good reduction for $\langle B;\Delta\rangle$?

The answer is YES for B finite (Theorem 4.3), but we expect that the answer is NO when the set B is not finite. However, future investigations can be made for finding some special classes of vector (n,m)-presentations $\langle B;\Delta\rangle$ where B is not finite, but Theorem 4.3 still holds.

2) Does the existence of a canonical ARS for the induced semigroup presentation $\langle B;\Lambda\rangle$ such that the unique normal forms of different elements from B^m are different, imply existence of a canonical ARS for $\langle B;\Delta\rangle$?

We expect that the answer is NO in this case, even when B is finite. This conclusions shall be proved and also, some special cases that fulfill YES as an answer might be searched.

3) The question if the construction in Theorem 4.1 (or some modified version) is possible for vector (n,m)-presentations of (n,m)-semigroups $\langle B;\Delta\rangle$ not satisfying the condition (4.1), remains open. Some of the problems that arise here are that some elements from B have to be identified in the (n,m)-semigroup, although they are different in the semigroup $\langle B;\Lambda\rangle$, or some elements of the form $(i,\mathbf{x})$ and $(j,\mathbf{y})$ for $i\neq j$ have to be identified.

The discussion above leads to the following question.

4) Is construction of a good reduction (or canonical ARS) possible in general case, i.e. for vector (n,m)-presentations $\langle B;\Delta\rangle$ permitting the corresponding set of induced binary relations Λ to contain pairs with length m?

The general answer is most probably NO (some counter examples might be found). Perhaps further classifications on Δ i.e. Λ shall be made, which would lead to appropriate conclusions and/or guides for future investigations.

Summing up, the existence of a good combinatorial description (or a canonical ARS) for $\langle B;\Lambda\rangle$ not necessarily implies existence of a good combinatorial description for $\langle B;\Delta\rangle$. Theorem 4.4 indicates that the above is true when B is finite. However,

5) Does the solution of the word problem for $\langle B;\Lambda\rangle$ imply solution to the word problem for $\langle B;\Delta\rangle$?

At this moment we do not have an answer to this question, although we have some indication that, in general, the answer is NO for B infinite and is YES for B finite.

APPLICATIONS IN VARIETIES OF (n,m)-SEMIGROUPS; GENERALIZATIONS THAT INCORPORATE NEW IDEAS

The introduction of vector (n,m)-presentations of (n,m)-semigroups has led to noticeable results for varieties of (n,m)-semigroups. The definition of a variety of (n,m)-semigroups was originally given in [2]. Vector varieties of (n,m)-semigroups and vector (n,m)-presentations in such varieties were introduced in [4]. Recent investigations were made in [35, 36, 37]. In [35] a direct description of the complete system of (n,m)-identities for a variety of (n,m)-semigroups is obtained. In [36] a characterization of vector varieties of (n,m)-semigroups is made. It is shown that the class of vector varieties of (n,m)-semigroups is a proper subset of the class of varieties of (n,m)-semigroups (when $m\geq 2$), and necessary and sufficient conditions for a variety of (n,m)-semigroups to be a vector variety are provided. In [37] a direct proof of Birkhoff's HSP theorem for varieties of (n,m)-semigroups is given. Moreover, a corresponding analog of this theorem for vector varieties of (n,m)-semigroups (when $m\geq 2$) is obtained.

The results exposed in this paper can be applied for appropriate classes of vector varieties of (n,m)-semigroups. Consequently, good reductions for vector (n,m)-presentations in some classes of vector varieties of (n,m)-semigroups might be constructed. This would lead to an existence of good descriptions for free objects in such (n,m)-varieties. There are various open questions concerning (vector) varieties of (n,m)-semigroups and numerous investigation possibilities within.

Vector valued semigroups provide a way of obtaining new languages. If we think of a binary operation as a process that from two information produces one information, then we can think of an $(m+k,m)$-operation as a process that from $m+k$ information, produces m information. Several authors (D. Dimovski, V. Manevska) have investigated formal vector valued languages and automata [15, 20]. The aim of our work, in a way, is to obtain a better understanding of these complicated languages. Afterwards, they might find possible application in ICT security systems. Quantum computers would additionally support this idea, and hopefully open new opportunities for incorporating these formal languages within ICT security systems. It is quite possible that the development of the combinatorial theory of (n, m)-semigroups has a bright future in front.

REFERENCES

[1] Gj. Čupona, Vector valued semigroups, *Semigroup Forum,* **26** (1983), pp. 65–74.

[2] Gj. Čupona, N. Celakoski, S. Markovski, D. Dimovski, Vector valued groupods, semigroups and groups, *Vector valued semigroups and groups,* Maced. Acad. of Sci. and Arts, Skopje, (1988), pp. 1–79.

[3] Gj. Čupona, D. Dimovski, On a class of vector valued groups, *Proc. Conf. Alg. and Logic, Zagreb,* (1984), pp. 29–37.

[4] Gj. Čupona, S. Markovski, D. Dimovski, B. Janeva, Introduction to the combinatorial theory of vector valued semigroups, *Vector valued semigroups and groups,* Maced. Acad. of Sci. and Arts, Skopje, (1988), pp. 141–184.

[5] Gj. Čupona, B. Trpenovski, Finitary and associative operations with neutral elements, *Bilten DMF NRM,* **12** (1961), pp. 15–24.

[6] D. Dimovski, Some existence conditions for vector valued groups, *God. zbor. na Mat. Fak., Skopje,* **(33-34)** (1982/83), pp. 99–103.

[7] D. Dimovski, Free vector valued semigroups, *Proc. Conf. Algebra and Logic,* Cetinje, (1986), pp. 55–62.

[8] D. Dimovski, Examples of $(2m + s, m)$-groups, *Contributions, Sec. Math. Tech. Sci.,* MASA, **VII**, 2 (1985), pp. 105–114.

[9] D. Dimovski, Free $(n+1, n)$-groups, *Vector valued semigroups and groups,* Maced. Acad. of Sci. and Arts, Skopje, (1988), pp. 103–122.

[10] D. Dimovski, S. Ilić, Free $(2m, m)$-groups, *Znanstvena Revija, Maribor,* (1990), pp. 29–40.

[11] D. Dimovski, B. Janeva, S. Ilić, Free (n, m)-groups, *Communications in Algebra,* **19** (3) (1991), pp. 965–979.

[12] D. Dimovski, Groups with unique product structures, *Jour. of Algebra,* **146** (1) (1992), pp. 205–209.

[13] D. Dimovski, On $(m + k, m)$-groups for $k < m$, *General Algebra and Discrete Mathematics,* K. Denecke, O. Luders (eds), Heldermann-Verlag, Berlin, 1995, pp. 93–99.

[14] D. Dimovski, Gj. Čupona, Injective vector valued semigroups, *Proc. VIII Int. Conf. Algebra and Logic, Novi Sad J. Math.,* **29** (2) (1999), pp. 149–161.

[15] D. Dimovski, V. Manevska, Vector Valued $(n+k, n)$-Formal Languages $(1 \leq k \leq n)$, *Proc.* 10$^{\text{th}}$ *Congr. of Yugosl. Math., Belgrade,* (2001), pp. 153–159.

[16] W.A. Dudek, K. Glazek, Around the Hosszú-Gluskin Theorem for n-ary groups, *Discrete Math.,* **308** (2008), pp. 4861–4876.

[17] M. Hosszú, On the explicit form of n-group operations, *Publ. Math., Debrecen,* **10** (1963), pp. 88–92.

[18] J.W. Klop, Term rewriting systems from Church-Rosser to Knuth-Bendix and beyond, Automata, languages and programming (ICALP'90), *Lecture Notes in Comput. Sci., Springer,* **443** (1990), pp. 350–369.

[19] J.W. Klop, *Term rewriting systems,* Handbook of logic in Computer Science, Claredon Press - Oxword, Vol. **2,** 1992, pp. 1–112.

[20] V. Manevska, D. Dimovski, $(3, 2)$-semigroups generated by $(3, 2)$-automata, *Math. Maced.,* (2004), pp. 65–81.

[21] M.H.A. Newman, On theories with a combinatorial definition of "equivalence", *Annals of Math.,* **43** (2) (1942), pp. 223–243.

[22] E.L. Post, Polyadic groups, Trans. *Amer. Math. Soc.,* **48** (1940), pp. 208–350.

[23] B.M. Schein, V.S. Trohimenko, Algebras of multiplace functions, *Semigroup Forum,* **17** (1979), pp. 1–64.

[24] B. Schweizer, A. Sklar, The Algebra of Multiplace Vector - Valued Functions, *Bull. Amer. Math. Soc.,* (1967), pp. 510–515.

[25] H.A. Thurston, Some properties of partly-associative operations, *Proc. Amer. Math. Soc.,* **5** (1954), pp. 487–497.

[26] K. Trenčevski, Some examples of fully commutative vector valued groups, *Contributions, Sec. Math. Tech. Sci.,* MASA, **IX**, 1/2, (1988), pp. 27–37.

[27] K. Trenčevski, D. Dimovski, *Complex commutative vector valued groups,* Maced. Acad. of Sci. and Arts, Skopje, (1992), pp. 1–117.

[28] K. Trenčevski, D. Dimovski, On the Afine and Projective Commutative $(m + k, m)$-Groups, *Journal of Algebra,* **240** (2001), pp. 338–365.

[29] Y. Zhu, On (n, m)-semigroups, *Semigroup Forum,* **84** (2) (2012), pp. 342–364.

[30] B. Trpenovski, Gj. Čupona, $[m,n]$-groupoids, *Bilten DMF SRM,* **21**(1970), pp. 19–29.

[31] I. Stojmenovska, *Combinatorial theory of vector valued semigroups,* PhD Thesis, UKIM, Skopje, 2011.

[32] I. Stojmenovska, D. Dimovski, Reductions for a class of presentations of (n, m)-semigroups, *Proc. Fourth Int. Sci. Conf. FMNS,* South-West University Neofit Rilski, Blagoevgrad, **1** (2011), pp. 70–76.

[33] D. Dimovski, I. Stojmenovska, Reductions for presentations of (n, m)-semigroups induced by reductions for presentations of binary semigroups, *Math. Maced.,* **7** (2009), pp. 45–56.

[34] D. Dimovski, I. Stojmenovska, Reductions for vector (n, m)-presentations of (n, m)-semigroups, *Semigroup Forum,* **86** (3) (2013), pp. 663–679.

[35] I. Stojmenovska, D. Dimovski, On varieties of (n, m)-semigroups, *International Journal of Algebra,* **6** (15) (2012), pp. 705–712.

[36] I. Stojmenovska, D. Dimovski, On vector varieties of (n, m)-semigroups, *International Journal of Algebra,* **12** (7) (2018), pp. 273–283.

[37] I. Stojmenovska, D. Dimovski, Birkhof's HSP Theorem for varieties of (n, m)-semigroups, *International Journal of Algebra,* **13** (2) (2019), pp. 51–58.

ЗА РЕДУКЦИИ НА ПРЕТСТАВУВАЊА НА ВЕКТОРСКО ВРЕДНОСНИ ПОЛУГРУПИ: ПРЕГЛЕД И ОТВОРЕНИ ПРОБЛЕМИ

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Посветено на нашиот сакан професор Ѓорѓи Чупона

Наоѓањето на комбинаторен опис за (n, m)-полугрупа зададена со (n, m)-претставување $\langle B; \Delta\rangle$ е прилично тешка задача и комплексен проблем. Поголемиот број добиени резултати се однесуваат на специјални класи претставувања на (n, m)-полугрупи наречени векторски (n, m)-претставувања на (n, m)-полугрупи. Истите индуцираат соодветни претставувања на бинарни полугрупи, поради што прашањето за постоење на добар комбинаторен опис на $\langle B; \Delta\rangle$ е тесно поврзано со прашањето за постоење на добар комбинаторен опис на соодветната индуцирана бинарна полугрупа $\langle B; \Lambda\rangle$. Правиме прегед на овие резултати, при што ги класифицираме условите под кои постоењето на добар комбинаторен опис за $\langle B; \Lambda\rangle$ имплицира решливост на проблемот на зборови во $\langle B; \Delta\rangle$. Дефинираме неколку отворени проблеми, посочуваме примена на добиените резултати во многуобразија (n, m)-полугрупи и даваме насоки за идни истражувања.

Клучни зборови: (n, m)-полугрупа, (n, m)-претставување, редукција, проблем на зборови

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ON A CLASS OF PRESENTATIONS IN VARIETIES OF VECTOR VALUED SEMIGROUPS

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To the memory of Professor Gjorgji Čupona, with deep respect and immense gratitude

We define a special class of (n, m)-semigroup presentations in vector varieties of (n, m)-semigroups and apply previously obtained results on existence of effective reductions within, under certain conditions. As a consequence, good combinatorial descriptions are provided.

Key words: (n, m)-semigroup, (n, m)-presentation, variety, reduction

INTRODUCTION

This work is a continuation of our results presented in [7, 8, 9]. In [10] we have discussed the word problem solvability for some classes of vector (n, m)-presentations. Here we try to apply some of those results for varieties of (n, m)-semigroups, in particular for some classes of vector varieties of (n, m)-semigroups. The introductory notions, basic definitions, and properties are incorporated in the review paper [10], that is our main reference paper. Bellow we annex few additional details necessary for the rest of the text.

- For an (n, m)-presentation of an (n, m)-semigroup $\langle B; \Delta\rangle$ (that is the factor (n, m)-semigroup $\boldsymbol{F(B)}/\overline{\Delta}$ where $\overline{\Delta}$ is the smallest congruence on $\boldsymbol{F(B)}$ such that $\Delta \subseteq \overline{\Delta}$ and $\boldsymbol{F(B)}/\overline{\Delta}$ is an (n, m)-semigroup), it can be easily shown that $\overline{\overline{\Delta}} = \overline{\Delta}$ ([2]).

- Two (n, m)-semigroup presentations $\langle B'; \Delta'\rangle$ and $\langle B''; \Delta''\rangle$ are strictly equivalent if $B' = B''$ and $\overline{\Delta'} = \overline{\Delta''}$. We use the notation $\langle B'; \Delta'\rangle \equiv \langle B''; \Delta''\rangle$ ([3]).

- Given a set of vector (n, m)-relations Δ, we will need to emphasize (in notation) the connection with its corresponding induced binary relations Λ. Thus, we allow elements from B to be represented as $(i, \mathbf{x})$ for some $\mathbf{x} \in B^m$ and $i \in \mathbb{N}_m$. Hence, given $u \in F(B)$ we will also use the notation $(i, u_1^{i-1} u u_{i+1}^m)$ where $i \in \mathbb{N}_m$ and $u_v \in F(B)$ $(v \in \mathbb{N}_m \backslash \{i\})$. In other words, we have the following notation definition:

$$u \in F(B) \Longleftrightarrow u = (i, \mathbf{x})$$

for some $i \in \mathbb{N}_m$, $\mathbf{x} = u_1^{m+sk}$, and $s \in \mathbb{N}_0$. (Note that, each element from $F(B)\backslash B$ remains to have a unique representation (i, u_1^{m+sk}) where $i \in \mathbb{N}_m$ and $s \geq 1$). Hence, for vector (n, m)-relations Δ and the corresponding induced binary relations Λ, we will also use the following notation

$$\Delta = \Lambda_\#$$

where

$$\Lambda_\# = \{((i, \mathbf{x}), (i, \mathbf{y})) \mid (\mathbf{x}, \mathbf{y}) \in \Lambda,\, i \in \mathbb{N}_m\}.$$

PRESENTATIONS IN VARIETIES OF (n, m)-SEMIGROUPS

The varieties of (n, m)-semigroups were defined in [2] and also explored in [3, 8, 9]. We recall basic definitions and properties necessary for the rest of the text.

If $\boldsymbol{F(}\mathbb{N}\boldsymbol{)}$ is a free poly-(n, m)-groupoid with a basis $\mathbb{N}$ and $\boldsymbol{Q} = (Q, h)$ is a poly-(n, m)-groupoid, for each $\tau \in F(\mathbb{N})$ there exists a smallest $t \in \mathbb{N}$ such that $\tau \in F(\mathbb{N}_t)$

and τ defines a t-ary operation on Q as follows:

i) If $\tau = j \in \mathbb{N}_t$ and $\mathbf{a} = a_1^t \in Q^t$ then $\tau(\mathbf{a}) = a_j$

ii) If $\tau = (i, \tau_1^{m+sk})$ and $\mathbf{a} = a_1^t \in Q^t$ then $\tau(\mathbf{a}) = h_i(\tau_1(\mathbf{a}) \dots \tau_{m+sk}(\mathbf{a}))$, assuming that $\tau_\nu(\mathbf{a})$ are already defined.

Let $\tau, \omega \in F(\mathbb{N})$. Then $\tau, \omega \in F(\mathbb{N}_t)$ for some $t \in \mathbb{N}$. A poly-(n,m)-groupoid $\boldsymbol{Q}$ satisfies the (n,m)-identity (τ, ω) (i.e. $\boldsymbol{Q} \models (\tau, \omega)$), if $\tau(\mathbf{a}) = \omega(\mathbf{a})$ for an arbitrary $\mathbf{a} = a_1^t \in Q^t$.

A class of (n,m)-semigroups $\mathcal{V}$ is a variety if and only if there exists a set of (n,m)-identities Θ such that $\boldsymbol{G} \models \Theta$ for every $\boldsymbol{G} \in \mathcal{V}$. This means that $\boldsymbol{G} \models (\tau, \omega)$, for every $(\tau, \omega) \in \Theta$ and every $\boldsymbol{G} \in \mathcal{V}$. We use the notation $\mathcal{V} = Var\Theta$.

In [8] we gave a description of the complete system of (n,m)-identities $\widehat{\Theta}$ for a variety $Var\Theta$. We also showed that $\psi_0(\boldsymbol{F(}\mathbb{N}\boldsymbol{)})/\widehat{\Theta}$ is a free object in $Var\Theta$ with basis $\mathbb{N}$ where ψ_0 is the reduction for $\langle \mathbb{N}; \emptyset \rangle$ (for more details on ψ_0, see [10]). In [9] we explored a special class of varieties of (n,m)-semigroups, called vector varieties of (n,m)-semigroups. They are originally defined in [3], as follows:

Let $p = m + sk, q = m + rk$, where $s, r \geq 0$ and let $(i_1^p, j_1^q) \in \mathbb{N}^+ \times \mathbb{N}^+$. An (n,m)-semigroup $\boldsymbol{G} = (G; g)$ satisfies the vector (n,m)-identity (i_1^p, j_1^q) (i.e. $\boldsymbol{G} \models (i_1^p, j_1^q)$), if $g(a_{i_1} \dots a_{i_p}) = g(a_{j_1} \dots a_{j_q})$ for an arbitrary $a_1^t \in G^t$, where $t = \max\limits_{\mu,\nu}\{i_\mu, j_\nu\}$.

Every vector (n,m)-identity (i_1^p, j_1^q) induces a set of (n,m)-identities $(i_1^p, j_1^q)_\# \subseteq \psi_0(F(\mathbb{N})) \times \psi_0(F(\mathbb{N}))$ defined by: $(i_1^p, j_1^q)_\# = \{((i, i_1^p), (i, j_1^q)) \,|\, i \in \mathbb{N}_m\}$, and moreover, $\boldsymbol{G} \models (i_1^p, j_1^q) \iff \boldsymbol{G} \models (i_1^p, j_1^q)_\#$. Consequently, if Θ' is a set of vector (n,m)-identities then it induces a set of (n,m)-identities $\Theta'_\#$, and, $\boldsymbol{G} \models \Theta' \iff \boldsymbol{G} \models \Theta'_\#$.

Definition 2.1 A variety of (n,m)-semigroups $\mathcal{V}$ is called a vector variety of (n,m)-semigroups, if there exists a set of vector (n,m)-identities $\Theta'_\#$ such that $\mathcal{V} = Var\Theta'_\#$.

In continuation we will define (n,m)-semigroup presentations in varieties of (n,m)-semigroups. The main idea arises from [3].

Let Θ be a set of (n,m)-identities and let $\boldsymbol{F(B)} = (F(B); f)$ be a free poly-(n,m)-groupoid with basis $B \neq \emptyset$. Every (n,m)-identity $(\tau, \omega) \in F(\mathbb{N}_t) \times F(\mathbb{N}_t)$ defines a relation on $F(B)$ given by

$$(\tau, \omega)(F(B)) = \{(\tau(u_1^t),\, \omega(u_1^t)) \,|\, u_1^t \in F(B)^t\}$$

Thus, Θ defines a corresponding set $\Theta(F(B)) \subseteq F(B) \times F(B)$ given by

$$\Theta(F(B)) = \bigcup_{(\tau,\omega)\in\Theta} (\tau, \omega)(F(B)) =$$
$$\{\, (\tau(u_1^t), \omega(u_1^t)) \,|\, (\tau, \omega) \in \Theta,$$
$$\tau, \omega \in F(\mathbb{N}_t),\, u_1^t \in F(B)^t,\, t \in \mathbb{N} \,\}.$$

Clearly, $\Theta(F(B))$ is a set of (n,m)-defining relations on B.

The following result is stated in [3], here we give its proof.

Proposition 2.1 *$\langle B; \Theta(F(B)) \rangle$ is a free object in $Var\Theta$ with basis B.*

Proof. Recall that $\langle B; \Theta\,(F(B)) \rangle = \boldsymbol{F(B)}/\overline{\Theta\,(F(B))}$ where $\overline{\Theta\,(F(B))}$ is the smallest congruence on $\boldsymbol{F(B)}$ such that $\Theta(F(B)) \subseteq \overline{\Theta\,(F(B))}$ and $\boldsymbol{F(B)}/\overline{\Theta\,(F(B))}$ is an (n,m)-semigroup. Let $(\tau, \omega) \in \Theta$. Then $\tau, \omega \in F(\mathbb{N}_t)$ for some $t \in \mathbb{N}$. For an arbitrary sequence $u_1^{\overline{\Theta(F(B))}}, \dots, u_t^{\overline{\Theta(F(B))}}$ from $F(B)/\overline{\Theta\,(F(B))}$, we have

$$\tau(u_1^{\overline{\Theta(F(B))}} \dots u_t^{\overline{\Theta(F(B))}}) = (\tau(u_1^t))^{\overline{\Theta(F(B))}} = (\omega(u_1^t))^{\overline{\Theta(F(B))}} = \omega(u_1^{\overline{\Theta(F(B))}} \dots u_t^{\overline{\Theta(F(B))}}).$$

Thus, $\boldsymbol{F(B)}/\overline{\Theta\,(F(B))} \models (\tau, \omega)$. Hence, $\boldsymbol{F(B)}/\overline{\Theta\,(F(B))} \models \Theta$ and therefore $\boldsymbol{F(B)}/\overline{\Theta\,(F(B))} \in Var\Theta$. It is clear that $\overline{\Theta\,(F(B))}$ is the smallest congruence on $\boldsymbol{F(B)}$ containing $\Theta\,(F(B))$ and such that $\boldsymbol{F(B)}/\overline{\Theta\,(F(B))} \in Var\Theta$, and thus we conclude that $\boldsymbol{F(B)}/\overline{\Theta\,(F(B))}$ is a free object in $Var\Theta$. Namely, for arbitraries $\boldsymbol{Q} \in Var\Theta$ and $\xi : B \to Q$, there is a unique homomorphic extension $\bar{\xi} : \boldsymbol{F(B)} \to \boldsymbol{Q}$ and moreover, $\boldsymbol{F(B)}/\ker\bar{\xi} \in Var\Theta$. The fact that $\overline{\Theta\,(F(B))}$ is the smallest congruence on $\boldsymbol{F(B)}$ such that $\boldsymbol{F(B)}/\overline{\Theta\,(F(B))}$ is in $Var\Theta$, implies that $\overline{\Theta\,(F(B))} \subseteq \ker\bar{\xi}$. Therefore, we define a map $\eta : F(B)/\overline{\Theta\,(F(B))} \to Q$, by: $\eta(u^{\overline{\Theta(F(B))}}) = \bar{\xi}(u)$. It is straightforward to check that η is a homomorphism, since $\bar{\xi}$ is a homomorphism, and

$\eta(nat(\overline{\Theta\,(F(B))})_{|B}) = \bar{\xi}_{|B} = \xi$. Also, η is unique, since $\bar{\xi}$ is unique. □

From now on, the congruence $\overline{\Theta\,(F(B))}$ will be denoted by $\overline{\Theta}$ and consequently, $\boldsymbol{F(B)}/\,\overline{\Theta\,(F(B))} = \boldsymbol{F(B)}/\,\overline{\Theta}$.

For a given $\Delta \subseteq F(B) \times F(B)$, we have $\Delta \cup \Theta\,(F(B)) \subseteq F(B) \times F(B)$, that is a set of (n,m)-defining relations on B, and thus $\langle B;\ \Delta \cup \Theta\,(F(B))\rangle$ is an (n,m)-presentation of an (n,m)-semigroup.

Definition 2.2 For given B, Θ, and Δ, we denote the (n,m)-semigroup presentation $\langle B;\ \Delta \cup \Theta\,(F(B))\rangle$ by $\langle B;\ \Delta;\ \Theta\rangle$, and we say that $\langle B;\ \Delta;\ \Theta\rangle$ is a presentation of an (n,m)-semigroup in the variety $Var\Theta$.

In particular, we define vector (n,m)-semigroup presentations in (vector) varieties of (n,m)-semigroups.

Definition 2.3 $\langle B;\ \Delta;\ \Theta\rangle$ is a vector presentation of an (n,m)-semigroup in $Var\Theta$, if $\langle B;\ \Delta\rangle$ and $\langle \mathbb{N};\ \Theta\rangle$ are vector (n,m)-presentations.

Thus, and by the notation given in the introduction part, given a vector (n,m)-semigroup presentation $\langle B;\ \Delta;\ \Theta\rangle$ in $Var\Theta$ we can also denote it as $\langle B;\ \Lambda;\ \Theta'\rangle$, where:

$\Lambda \subseteq B^+ \times B^+$ and $\Delta = \Lambda_{\#}$;

$\Theta' \subseteq \mathbb{N}^+ \times \mathbb{N}^+$ and $\Theta = \Theta'_{\#}$.

Given $\langle B;\ \Lambda;\ \Theta'\rangle$, the set of vector (n,m)-identities $\Theta' \subseteq \mathbb{N}^+ \times \mathbb{N}^+$ induces a set $\Theta'(B) \subseteq B^+ \times B^+$ defined by:

$(a_1^p, c_1^q) \in \Theta'(B)$ if there exist $(i_1^p, j_1^q) \in \Theta'$ and a sequence $b_1, b_2, \ldots \in B$ such that

$a_\mu = b_{i_\mu}$, $\mu \in \mathbb{N}_p$ and $c_\upsilon = b_{j_\upsilon}$, $\upsilon \in \mathbb{N}_q$.

In other words,

$$\Theta'(B) = \{(b_{i_1} \ldots b_{i_p},\ b_{j_1} \ldots b_{j_q}) \mid (i_1^p, j_1^q) \in \Theta',\ b^t \in B^t,\ t = \max_{\mu,\upsilon}\{i_\mu, j_\upsilon\}\}.$$

Now, $\Lambda \cup \Theta'(B) \subseteq B^+ \times B^+$ is a set of vector (n,m)-relations on B, and thus $\langle B; \Lambda \cup \Theta'(B)\rangle$ is a vector (n,m)-presentation of an (n,m)-semigroup. But, $\langle B; \Lambda \cup \Theta'(B)\rangle$ is not in $Var\Theta'_{\#}$ in general case.

Example 2.1. Let $n = 3$, $m = 2$, $B = \{a, b\}$, $\Lambda = \emptyset$, and let Θ' be a set of $(3,2)$-identities defined by:

$\Theta' = \{(l^3, l^2)\}$ for some $l \in \mathbb{N}$, i.e.

$$\Theta'_{\#} = \{((1,lll),(1,ll)),((2,lll),(2,ll))\} = \{((1,lll),l),((2,lll),l)\}.$$

We have that $\langle a, b;\ \Theta'\rangle = \langle a, b;\ \Theta'_{\#}\rangle$ is a $(3,2)$-semigroup presentation in $Var\Theta'_{\#}$. Moreover, the $(3,2)$-semigroup $\langle a, b;\ \Theta'_{\#}\rangle = \boldsymbol{F(a,b)}/\,\overline{\Theta'_{\#}\,(F(a,b))}$ is a free object in $Var\Theta'_{\#}$ with basis $\{a, b\}$. On the other hand, the $(3,2)$-semigroup presentation $\langle B;\ \Theta'(B)\rangle = \langle a, b;\ \Theta'(a,b)\rangle$ represents the $(3,2)$-semigroup $\boldsymbol{F(a,b)}/\,\overline{(\Theta'(a,b))_{\#}}$. It is easy to see that $(\Theta'(a,b))_{\#} \subseteq \Theta'_{\#}\,(F(a,b))$ and thus $\overline{(\Theta'(a,b))_{\#}} \subseteq \overline{\Theta'_{\#}\,(F(a,b))}$. Consequently, if two elements are equal in $\langle a, b;\ \Theta'(a,b)\rangle$, they are equal in $\langle a, b;\ \Theta'\rangle$ as well, The opposite is not true. For example, $(2,\ (1,\ aba)(1,\ aba)(1,\ aba)) = (1,\ aba)$ in $\langle a, b;\ \Theta'\rangle$ but $(2,\ (1,\ aba)(1,\ aba)(1,\ aba)) \neq (1,\ aba)$ in $\langle a, b;\ \Theta'(a,b)\rangle$. We conclude that $\langle a, b;\ \Theta'(a,b)\rangle \notin Var\Theta'_{\#}$. □

Proposition 2.2
$\langle B;\ \Lambda;\ \Theta'\rangle \equiv \langle B; \Lambda \cup \Theta'(B)\rangle$ if and only if $\langle B; \Lambda \cup \Theta'(B)\rangle \in Var\Theta'_{\#}$.

Proof. ($\Rightarrow$). Straightforward.
($\Leftarrow$). It is easy to notice that

$$(\Lambda_{\#} \cup \Theta'(B)_{\#}) \subseteq (\Lambda_{\#} \cup \Theta'_{\#}(F(B))),$$

and thus $\overline{\Lambda_{\#} \cup \Theta'(B)_{\#}} \subseteq \overline{\Lambda_{\#} \cup \Theta'_{\#}(F(B))}$.
Since $\boldsymbol{F(B)}/\overline{\Lambda_{\#} \cup \Theta'(B)_{\#}} \in Var\Theta'_{\#}$, it follows that for $i \in \mathbb{N}_m$, for an (n,m)-identity $(i_1^{p'}, j_1^{q'}) \in \Theta'$, and for a sequence u_1^t from $F(B)$, where $t = \max_{\mu,\upsilon}\{i_\mu, j_\upsilon\}$:

$$f_i(u_{i_1}^{\overline{\Lambda_{\#} \cup \Theta'(B)_{\#}}} \ldots u_{i_{p'}}^{\overline{\Lambda_{\#} \cup \Theta'(B)_{\#}}}) = f_i(u_{j_1}^{\overline{\Lambda_{\#} \cup \Theta'(B)_{\#}}} \ldots u_{j_{q'}}^{\overline{\Lambda_{\#} \cup \Theta'(B)_{\#}}}),$$

i.e. $(f_i(u_{i_1} \ldots u_{i_{p'}}))^{\overline{\Lambda_{\#} \cup \Theta'(B)_{\#}}} = (f_i(u_{j_1} \ldots u_{j_{q'}}))^{\overline{\Lambda_{\#} \cup \Theta'(B)_{\#}}}$.

This implies that

$$\left((i, u_{i_1}^{i_{p'}}), (i, u_{j_1}^{j_{q'}})\right) \in \overline{\Lambda_{\#} \cup \Theta'(B)_{\#}}$$

and thus

$$\Theta'_{\#}(F(B)) \subseteq \overline{\Lambda_{\#} \cup \Theta'(B)_{\#}}.$$

Consequently,

$$\Lambda_{\#} \cup \Theta'_{\#}(F(B)) \subseteq \overline{\Lambda_{\#} \cup \Theta'(B)_{\#}},$$

and moreover,

$$\overline{\Lambda_{\#} \cup \Theta'_{\#}(F(B))} \subseteq \overline{\Lambda_{\#} \cup \Theta'(B)_{\#}}.$$

Hence, $\langle B;\ \Lambda;\ \Theta'\rangle \equiv \langle B; \Lambda \cup \Theta'(B)\rangle$. □

Consider now, vector (n,m)-presentations of type $\langle B; \Lambda \cup \Theta'(B)\rangle$.

Since $\langle B; \Lambda \cup \Theta'(B)\rangle$ is a vector (n,m)-presentation of an (n,m)-semigroup, it induces a corresponding binary semigroup presentation, for which we can apply Theorem 4.1, Theorem 4.2, Theorem 4.3, Theorem 4.4 from [10]. As a consequence, and providing that Proposition 2.2 is satisfied, we would get good combinatorial descriptions for $\langle B;\, \Lambda;\, \Theta'\rangle$, that are objects in $Var\Theta'_{\#}$. Moreover, we would have word problem solvability for those vector (n,m)-semigroup presentations in such varieties.

REFERENCES

[1] Gj. Čupona, Vector valued semigroups, *Semigroup Forum,* **26** (1983), pp. 65–74.

[2] Gj. Čupona, N. Celakoski, S. Markovski, D. Dimovski, Vector valued groupods, semigroups and groups, *Vector valued semigroups and groups,* Maced. Acad. of Sci. and Arts, Skopje, (1988), pp. 1–79.

[3] Gj. Čupona, S. Markovski, D. Dimovski, B. Janeva, Introduction to the combinatorial theory of vector valued semigroups, *Vector valued semigroups and groups,* Maced. Acad. of Sci. and Arts, Skopje, (1988), pp. 141–184.

[4] D. Dimovski, Free vector valued semigroups, *Proc. Conf. Algebra and Logic,* Cetinje, (1986), pp. 55–62.

[5] D. Dimovski, Gj. Čupona, Injective vector valued semigroups, *Proc. VIII Int. Conf. Algebra and Logic, Novi Sad J. Math.,* **29** (2) (1999), pp. 149–161.

[6] D. Dimovski, B. Janeva, S. Ilić, Free (n,m)-groups, *Communications in Algebra,* **19** (3) (1991), pp. 965–979.

[7] D. Dimovski, I. Stojmenovska, Reductions for vector (n,m)-presentations of (n,m)-semigroups, *Semigroup Forum,* **86** (3) (2013), pp. 663–679.

[8] I. Stojmenovska, D. Dimovski, On varieties of (n,m)-semigroups, *International Journal of Algebra,* **6** (15) (2012), pp. 705–712.

[9] I. Stojmenovska, D. Dimovski, On vector varieties of (n,m)-semigroups, *International Journal of Algebra,* **12** (7) (2018), pp. 273–283.

[10] I. Stojmenovska, D. Dimovski, On reductions for presentations of vector valued semigroups: overview and open problems. To appear in this vollume.

ЗА ЕДНА КЛАСА ПРЕТСТАВУВАЊА ВО МНОГУОБРАЗИЈА ВЕКТОРСКО ВРЕДНОСНИ ПОЛУГРУПИ

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Во спомен на професор Ѓорѓи Чупона, со длабока почит и огромна благодарност

Дефинираме специјална класа векторски (n,m)-претставувања во векторски многуобразија (n,m)-полугрупи, каде аплицираме претходно добиени резултати за постоење на ефективни редукции, под одредени услови. Како последица, се добиваат добри комбинаторни описи на разгледуваните објекти.

Клучни зборови: (n,m)-полугрупа, (n,m)-претставување, (n,m)-многуобразие, редукција

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INVERSE SAMPLING DESIGNS

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Dedicated to Professor Gjorgji Čupona

Using the algebraic definition of a sampling design introduced in [6], and the notion of quotient sampling designs described in [5] and [6] we present the definition of inverse sampling designs and examine some of their properties.

Key words: semigroup, free semigroup, epimorphism, sampling design

INTRODUCTION

In the sampling theory, as a part of mathematical statistics, that has been developed for several decades, one can find different approaches in selecting a sample from a population. The discrete and very often finite nature of population that is of interest in the theory of sampling design, enables use of finite algebraic structures in research in this area of statistics. In [6] we have examined the algebraic structure of sampling designs, gave a unified formal definition of the notion of sampling design that opened the way of construction of new interesting designs with some better properties in terms of their use in statistical inference. In [5] we have shown how to construct a quotient design of a given sampling design. In this paper we present the results about the opposite task, namely, we construct inverse sampling designs that can be associated to a given sampling design.

Further on, when it is clear from the contest, we will use only the word design instead of sampling design.

At the beginning we present some preliminaries. In sections two and three we state the unified definition of a sampling design as an algebraic structure and the definition of a quotient design, give some examples and state some already published results about quotient designs. In section four a construction and characterization of inverse designs is given.

Let $B = \{b_1, \cdots b_N\}$ be a finite set (called population), S=S(B) be a semigroup generated by B, and U=U(B) be a free semigroup generated by B. The elements of U(B) will be denoted by σ, τ, ω, … and the elements of the semigroup S(B) by s, t, u, ….

Let $\sigma \in U, \sigma = b_1 \cdots b_n$, for $b_i \in B$. For a given $b \in B$, we say that $b \in \sigma$, if $b = b_i$ for some $1 \leq i \leq n$. The *length* L(σ) of σ is n. We define the *content* $C(\sigma)$ of $\sigma \in U$, by

$$C(\sigma) = \{b \mid b \in \sigma\}.$$

If $S(B)$ is a semigroup generated by B, then there exists a unique homomorphism (which is an epimorphism) $\psi: U(B) \to S(B)$ such that $\psi(b) = b$ for each $b \in B$ ([1]). From now on we will use the symbol ψ only for this epimorphism.

SAMPLE AS AN ELEMENT OF A SEMIGROUP

In this section we give the definition of a sampling design via semigroups and give some examples that show how some known sampling designs can be represented in terms of this definition.

Let $B = \{b_1, \cdots, b_N\}$ be an identifiable population and S(B) a semigroup generated by B.

Definition 2.1 A *sampling design* over the population B and the semigroup S is an ordered triple $\boldsymbol{P} = (B, S, p)$, where $p: S(B) \to \mathbb{R}$ is a real function such that:

i) For each $s \in S, p(s) \geq 0$; and
ii) $\sum_{s \in S(B)} p(s) = 1$.

The semigroup $S(B)$ is called a *sampling set* and the function p - a *design function***.** The elements of $S(B)$ are called S – *samples* over B, i.e., samples over B in the semigroup S.

A *carrier* of the design $\boldsymbol{P}$ is the set

$$S_p = \{s | s \in S(B), p(s) > 0\}.$$

A unit $b \in B$ *belongs* to a sample $s \in S$, denoted by $b \in s$, if $s = a_1 \cdots a_n$ for $a_1, \cdots, a_n \in B$ and there is an i such that $1 \leq i \leq n$ and $b = a_i$. In other words $b \in s$ if and only if there is a $\sigma \in U$, such that $b \in \sigma$ and $\psi(\sigma) = s$.

A sampling design $P = (B, S, p)$ is called a *regular* design if for each $b \in B$, there is an $s \in S_p$ such that $b \in s$.

We say that a sampling design $\boldsymbol{P} = (B, S, p)$ is *finite* design if the carrier of $\boldsymbol{P}$, S_p is a finite set.

The *content* $C(s)$ of s, is defined by

$$C(s) = \{C(\sigma) | \sigma \in U, \psi(\sigma) = s\}.$$

By this definition we have that if $b \in B$, $s \in S$, then $b \in s$ if and only if $b \in \cup \{C | C \in C(s)\}$.

The *length* $L(s)$ of s, is defined by

$$L(s) = \{n | \ n = L(\sigma), \psi(\sigma) = s\}.$$

In other words, the length of s is the set of natural numbers that are lengths of the representations of s as a product, i.e., that are lengths of all $\sigma \in U$, such that $\psi(\sigma) = s$.

We say that a pair (B, S) satisfies the condition for *uniqueness of content* (or *length*) if and only if $C(s)$ (or $L(s)$) is a set with one element, for each $s \in S(B)$.

The following examples illustrate the representation of different sampling designs, known in literature, dealing with sampling designs, in terms of Definition 2.1.

Example 2.1. In [2] a sample is defined as a finite sequence of units of a population with replications – *ordered sampling design with replications.* It can be represented by a design $\boldsymbol{P} = (B, U, p)$ where U is a free semigroup generated by B, whose elements (samples) are finite ordered sequences of B with replication. By the definition, it follows that the pair (B, U) satisfies the condition for uniqueness of contents and the condition for uniqueness of length. The same representation $\boldsymbol{P} = (B, U, p)$ is valid for an *ordered sampling design with replications of fixed length,* where if $p(s) > 0$, the length of s is some fixed number m.

Example 2.2. In [3] a sample is defined as a finite sequence of units of a population without replications – *ordered sampling design without replications*. This design is of the form $\boldsymbol{P} = (B, S, p)$, where S is the semigroup generated by B in which the following identities hold for each $x, y \in S$:

$$x^2 = x \quad \text{and} \quad xyx = xy.$$

The condition for uniqueness of contents is satisfied but not the condition for uniqueness of length, although there is a unique canonical representation of each element of S as a product of units and can be used for definition of unique content and length.

Example 2.3. In [4] a sample is defined as a subset of B – *unordered sampling design without replication*. According to our definition, this design can be represented by $(B, M(B), p)$ where $M(B)$ is a free semi-lattice generated by B, i.e. the semigroup where, for each $x, y \in B$, the following identities hold

$$x^2 = x \text{ and } xy = yx.$$

The condition for uniqueness of content is satisfied but the condition for uniqueness of length in general is not satisfied, although as in the previous example there is a unique canonical representation for each element of $M(B)$ that can be used for definition of unique content and length.

Example 2.4. A sampling design where the sample is defined as a multi subset of B, is called, *an unordered sampling design with replications*. By Definition 2.1, a design of this type over a population B is of the form (B, N, p) where $N = N(B)$ is a free commutative semigroup generated by B, i.e. the semigroup in which the following identity holds for each $x, y \in B$:

$$xy = yx$$

Both conditions for uniqueness of content and length are satisfied.

Example 2.5. We give an example of a design that doesn't satisfy neither the condition for uniqueness of content nor the condition for uniqueness of length. Such a design is the design (B, S, p) where S is a semigroup in which the following identity

$$xyz = xuz$$

holds for each $x, y, z, u \in S$.

QUOTIENT DESIGNS

In this section we give the definition of quotient designs introduced in [5] and state some properties which are discussed and proven there.

Assume that $S = S(B)$ and $S' = S'(B')$ are semigroups generated by finite populations B and B', and $|B| = N,\ |B'| = N'$ with $N' \leq N$.

Theorem 3.1 *Let $\boldsymbol{P} = (B, S, p)$ be a sampling design and let $\varphi : S \to S'$ be an epimorphism such that $\varphi(B) = B'$. If $p' = p_\varphi : S' \to \mathbb{R}$ is defined by*

$$p'(s') = \sum_{s\in\varphi^{-1}(s')} p(s) \ \text{ for each } s' \in S',$$

then, $\boldsymbol{P}' = (B', S', p')$ is a sampling design such that $S'_{p'} = \varphi(S_p)$. ■

We say that $\boldsymbol{P'}$ is a *quotient design* of the design $\boldsymbol{P}$ by the epimorphism φ, and denote it by $\boldsymbol{P_\varphi}$. In the same sense, we say that $\boldsymbol{P}$ is an *φ-inverse design* (or just *inverse design*) of the design $\boldsymbol{P_\varphi}$.

In the above theorem and further on, for abbreviation, we use $\varphi^{-1}(s)$ instead of $\varphi^{-1}(\{s\})$.

Theorem 3.2 *Any design (B, S, p') is a quotient design by some epimorphism ψ of some design (B, U, p).* ■

Proposition 3.3 *Quotient design of a regular design is a regular design.* ■

Proposition 3.4 *Quotient design of a finite design is a finite design.* ■

Proposition 3.5 *For any design $\boldsymbol{P} = (B, S, p)$ there is a quotient design $\boldsymbol{P_\varphi}$ which is regular and finite.* ■

Example 3.1. Let $S' = \{1\}$ be the semigroup with one element, and $B' = S' = \{1\}$. Then there is a unique design $\boldsymbol{P}' = (B', S', p')$ for which $p'(1) = 1$. The design $\boldsymbol{P}'$ is regular and finite and is a quotient design of any design $\boldsymbol{P}$.

Proposition 3.6 *Any finite design which is not regular has a quotient design which is not regular.* ■

Proposition 3.7 *Any regular design which is not finite has a quotient design that is not finite.* ■

INVERSE SAMPLING DESIGNS

In the previous section we gave a construction of a quotient design $\boldsymbol{P}' = (B', S', p')$ for a given design $\boldsymbol{P} = (B, S, p)$ and epimorphism $\varphi: S \to S'$, and called the design $\boldsymbol{P}$ a φ-inverse design of $\boldsymbol{P}'$. In this section we will look at the opposite task, i.e., for a given design and given epimorphism, we will construct inverse designs.

Theorem 4.1 *Assume that $S = S(B)$ and $S' = S'(B')$ are semigroups generated by finite populations B and B', $|B| = N$, $|B'| = N'$ with $N' \leq N$, and $\varphi: S \to S'$ is an epimorphism such that $\varphi(B) = B'$.*

Let $\boldsymbol{P}' = (B', S', p')$ be a sampling design and let for each $s' \in S'$,

$$p_{s'} : \varphi^{-1}(s') \to \mathbb{R}$$

be a function, such that:

a) $p_{s'}(s) \geq 0$ for each $s \in \varphi^{-1}(s')$;

b) $\sum_{s\in\varphi^{-1}(s')} p_{s'}(s) = p'(s')$.

If the function $p: S \to \mathbb{R}$ is defined by

$$p(s) = p_{\varphi(s)}(s),$$

then: $\boldsymbol{P} = (B, S, p)$ is a φ-inverse design of $\boldsymbol{P}'$; $S_p \subseteq \varphi^{-1}(S'_{p'})$; and $\boldsymbol{P}' = \boldsymbol{P_\varphi}$.

Proof. First of all, since $\varphi(s)$ is completely determined by s, $p(s)$ is well defined real number, and it is clear that $p(s) \geq 0$. On the other hand

$$\sum_{s\in S} p(s) = \sum_{s\in S} p_{\varphi(s)}(s) = \sum_{s\in\varphi^{-1}(s')} \sum_{s'\in S'} p_{s'}(s)$$

$$= \sum_{s'\in S'} \sum_{s\in\varphi^{-1}(s')} p_{s'}(s)$$

$$= \sum_{s'\in S'} p'(s') = 1.$$

So, $\boldsymbol{P} = (B, S, p)$ is a design over $S = S(B)$ and

$$p'(s') = \sum_{s\in\varphi^{-1}(s')} p_{s'}(s) = \sum_{s\in\varphi^{-1}(s')} p(s).$$

This implies that $\boldsymbol{P}'$ is a quotient design of $\boldsymbol{P}$ by φ, and so, $\boldsymbol{P}$ is φ-inverse design of $\boldsymbol{P}'$. ■

Let us note that if $s'_1, s'_2 \in S'$ and $s \in S$ are such that $s \in \varphi^{-1}(s'_1) \cap \varphi^{-1}(s'_2)$, then $s'_1 = s'_2$, meaning that $p: S \to \mathbb{R}$ is well defined. Also, if we consider the family of all functions $p_{s'}: \varphi^{-1}(s') \to \mathbb{R}$, which satisfy the conditions a) and b) in Theorem 4.1, we will obtain the family of all designs $\boldsymbol{P} = (B, S, p)$ that are φ-inverse of $\boldsymbol{P}'$. Particularly, if $\boldsymbol{P}' = \boldsymbol{P_\varphi}$, putting $p_{\varphi(s)}(s) = p(s)$, we will get the initial design $\boldsymbol{P}$.

We should emphasise that if $s' \in S'$ is such that $p'(s') = 0$, i.e., $s' \notin S'_{p'}$, then $p_{s'}(s) = 0$, for all $s \in \varphi^{-1}(s')$. Nevertheless, it is possible to have $p(s) = p_{\varphi(s)}(s) = 0$ for some $\varphi(s) \in S'_{p'}$.

The first part of the next theorem follows from the Propositions 3.5 –3.7, the Example 3.1 and the Theorem 4.1.

Theorem 4.2 *(i) It is possible that an inverse design of a: a) finite, b) regular, or c) finite and regular design, does not have the same property.*

(ii) A design $\boldsymbol{P}'$ has some of the properties a), b), or c) if and only if there is an inverse design $\boldsymbol{P}$ of $\boldsymbol{P}'$ that has the same property.

Proof. From the Theorem 4.1 and the Propositions 3.3 and 3.4 it follows that if there is an inverse design $\boldsymbol{P}$ of $\boldsymbol{P'}$ that has some of the properties *a*), *b*) or *c*), then the design $\boldsymbol{P'}$ has the same property.

To prove the other direction of part (*ii*) of the theorem, for $\boldsymbol{P'}$ finite or regular, we give a construction of an inverse design that is finite and inverse design that is regular and an inverse design that is finite and regular.

Let $\boldsymbol{P'}$ be a finite design. We will construct a finite inverse design of $\boldsymbol{P'}$.

For all $s' \in S'_{p'}$ we choose $s_1, \cdots, s_{k(s')} \in \varphi^{-1}(s')$ and real numbers $p_{1,s'}, \cdots p_{k(s'),s'} > 0$ such that

$$\sum_{i=1}^{k(s')} p_{i,s'} = p'(s).$$

Then the corresponding inverse design $\boldsymbol{P}$ is finite. Let us note that if we are varying $s_1, \cdots, s_{k(s')}$ and the numbers $p_{i,s'}$ over all possible values, we will get all possible finite inverse designs of $\boldsymbol{P'}$.

Let us assume now that $\boldsymbol{P'}$ is a regular design. Then, there is a finite subset A' of $S'_{p'}$ such that for each $b' \in B'$, there is $s' \in A'$ such that $b' \in s'$. (We can assume that A' is the minimal set with this property, which will mean that $|A'| \leq N$, but for the following discussion this is irrelevant.) Then, for $s' \in A'$ let $a'_1, \cdots a'_{k(s')}$ be the elements of B' for which $a'_t \in s'$. We are looking at all units $b \in B$, for which $\varphi(b) = a'_t$ for some *t*. For each *b* with this property, we choose $s_b \in S$, such that $\varphi(s_b) = s'$ and $b \in s_b$. (This is possible since, from $a'_t \in s'$ it follows that $s' = t'a'_t q'$, where $t', q' \in S' \cup \{\lambda\}$, and λ is the empty sequence. So, the s_b we are looking for is $s_b = tbq$, for $t \in \varphi^{-1}(t')$ and $q \in \varphi^{-1}(q')$.) By $A(s')$ we denote the set of all such s_b.

Finally, we choose a function $f_{s'}: A(s') \to \mathbb{R}$ such that $p'(s') = \sum_{s \in A(s')} f_{s'}(s)$ and for each $s \in S'$, $f_{s'}(s) > 0$. Such a function $f_{s'}$ exists, since $p'(s') > 0$ and $A(s')$ are finite. For example, we can define $f_{s'}(s) = p'(s')/|A(s')|$. Then, the function $p_{s'}: \varphi^{-1}(s') \to \mathbb{R}$ is defined by

$$p_{s'}(s) = \begin{cases} f_{s'}(s) & \text{for } s \in A(s') \\ 0 & \text{otherwise} \end{cases}.$$

Any inverse design $\boldsymbol{P}$ obtained in this way is regular.

Note that by taking different choices for the sets $A(s')$ as well as different functions $f_{s'}$, we will get different regular inverse designs.

At the end, if $\boldsymbol{P'}$ is finite and regular, we can take $A' = S'_{p'}$, so, any regular inverse design $\boldsymbol{P}$, obtained by the previous discussion, is finite too. ∎

Proposition 4.3 *For arbitrary design* $\boldsymbol{P'}(B', S', p')$ *there is a* φ*-inverse design* $\boldsymbol{P}(B, S, p)$ *of* $\boldsymbol{P'}$ *such that the function* $\tilde{\varphi}: S_p \to S'_{p'}$*, induced by* φ*, is a bijection. The design* $\boldsymbol{P}$ *is finite if and only if* $\boldsymbol{P'}$ *is finite.*

Proof. We obtain such a design if for each $s' \in S'_{p'}$ we choose exactly one $s \in \varphi^{-1}(s')$ and put $p_{s'}(s) = p'(s')$ and $p_{s'}(t) = 0$ for any other $t \in \varphi^{-1}(s'), t \neq s$. If $\boldsymbol{P'}$ is finite, $\boldsymbol{P}$ is finite too, since $|S_p| = |S'_{p'}|$. ∎

With the next example we show that the last conclusion of the previous proposition does not hold for regular designs.

Example 4.1. Let $B = \{b_1, \cdots, b_N\}$, $U = U(B)$, $B' = \{b_1, \cdots, b_{N-1}\}$, $U' = U'(B')$ and let $\varphi: U \to U'$ be the epimorphism generated by $b_i \longmapsto b_i$ for $i \leq N-1$ and $b_N \longmapsto b_{N-1}$. Let $\boldsymbol{P'} = (B', S', p')$ be a regular design. Note that U' is a subsemigroup of U, so $U'_{p'}$ is a subset of U. If we take $U_p = U'_{p'}$, $p(s) = p'(s)$, for $s \in U_p$ and $p(s) = 0$ for $s \notin U_p$, we obtain an inverse design $\boldsymbol{P}$ of the design $\boldsymbol{P'}$ which is not regular *even* though $\tilde{\varphi}: U_p \to U'_{p'}$ is a bijection.

The validity of the next proposition is a consequence of the Theorem 4.2.

Proposition 4.4 *There is a unique* φ*-inverse design of a given design* $\boldsymbol{P'} = (B', S', p')$ *if and only if* $\varphi^{-1}(s')$ *has only one element for each* $s' \in S'_{p'}$*. If this condition is not satisfied, then there are infinitely many* φ*-inverse designs of the design* $\boldsymbol{P'}$*.* ∎

Proposition 4.5 *Any* φ*-inverse design of a finite design* $\boldsymbol{P'}$ *is finite if and only if* $\varphi^{-1}(s')$ *is finite for all* $s' \in S'_{p'}$*.*

Proof. Let $s'_0 \in S'_{p'}$ be such that $\varphi^{-1}(s'_0)$ is an infinite set and let $A = \{s_1, \cdots, s_n, \cdots\} \subseteq \varphi^{-1}(s')$ be such that for $i \neq j, s_i \neq s_j$. We choose a sequence of positive real numbers $p_1, \cdots, p_n, \cdots$ such that

$$\sum_{n=1}^{\infty} p_n = p'(s'_0).$$

If $\boldsymbol{P}$ is a φ-inverse design of $\boldsymbol{P'}$ such that $p(s_i) = p_i$, then $\boldsymbol{P}$ is not finite since $A \subseteq S_p$. ∎

For a similar characterisation of regular designs as the previous property, we need to introduce the following notion.

Let $\boldsymbol{P} = (B, S, p)$ be a sampling design. We say that a subset $T \subseteq S$ is a *regular subset* of *S* if for each $b \in B$, there is a $t \in T$, such that $b \in t$.

The subset $T \subseteq S$ is a *minimal regular subset* of *S* if no other proper subset of *T* is regular.

If T' is minimal regular subset of S' such that $T' \subseteq S'_{p'}$ and $T \subseteq \varphi^{-1}(T')$ is such that for each $s' \in T'$, $|T \cap \varphi^{-1}(s')| = 1$, then *T* is regular subset of S_p.

Proposition 4.6 *A φ-inverse design of a regular design* $\boldsymbol{P}' = (B', S', p')$ *is regular if and only if any subset* T *of* $\varphi^{-1}(S'_{p'})$ *such that for each* $s' \in S'_{p'}$, $|T \cap \varphi^{-1}(s')| = 1$, *is regular in* S. ■

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My interest in algebraic structures dates back to the first year of my studies when my professor of Elementary Algebra was Professor Gjorgji Čupona. Since then, he has had a great impact on my academic and professional development.

His open mind, wide interest in different mathematical disciplines, the knowledge he unreservedly and skillfully transmitted, as well as his attitude towards his students and collaborators have been a valuable example and inspiration in my teaching and scientific work. His memorable lectures, the blackboard that at the end of the lectures looked like a carefully written part of a textbook, and mandatory consultations for all of his students at 7:30 am before the start of classes at 8:15, sparked my scientific interest in the mathematical disciplines.

When I started working on my doctoral dissertation and we discussed the problems of my interest in mathematical statistics, his suggestion was to try to apply algebraic structures in sampling theory believing that in this way many of the questions of interest could be answered more easily.

I fill lucky and grateful that I had Professor Čupona as my teacher, academic and scientific advisor and a valuable friend.

REFERENCES

[1] Ѓ. Чупона, Б. Трпеновски, *Предавања по алгебра*, книга II, Скопје, (1973)

[2] C. Claes-Magnus, *Foundations of Inference in Survey Sampling*, John & Sons, Inc. (1977)

[3] A. S. Hedajat, B.K. Sinha, *Design and inference in Finite Population Sampling*, Wiley series in probability and mathematical statistics, (1990)

[4] Jugoslav Hajec, *Sampling From a Finite Population*, Marcel Dekker, Inc. New York, (1981)

[5] Ž. Popeska, Quotient sampling design, *Contributions, Sec. Math. Tech. Sci.*, MASA, **XV**, 1 (1994), pp. 37–48

[6] Ж. Попеска, *Алгебарско комбинаторни методи во теорија на примерок и експериментален дизајн*, Докторска дисертација, ПМФ, УКИМ Скопје, (1998)

ИНВЕРЗНИ ДИЗАЈНИ НА ПРИМЕРОК

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Користејќи ја дефиниција на примерок во термини на алгебарски структури, воведена во [6], како и поимот за фактор план на примерок опишан во [5] и [6] дефинираме инверзен план на примерок и ги испитуваме некои од својствата кои ги имаат овие планови.

Клучни зборови: полугрупи, слободна полугрупа, епиморфизам, план на примерок.

Мојот интерес за алгебрарските структури потекнува уште од првата година на моите студии по математика, кога професор по елементарна алгебра ми беше професорот Ѓорѓи Чупуна. Од тогаш па се до денес, тој имаше големо влијание и беше дел од мојот научен и професионален развој. Неговата сестраност како математичар, знаењето кое безрезервно и умешно го пренесуваше, како и неговиот однос кон студентите и соработниците претставуваа пример и инспирација во мојата наставна и научна работа. Неговите незаборавни предавања, таблата која на крајот на часовите изгледаше како грижливо напишан дел од учебник и задолжителните консултации со сите студенти во вторник во 7:30, пред почетокот на часовите во 8:15 во Математичкиот амфитеатар на ПМФ, го побудија мојот научен интерес во математичките дисциплини. Кога почнав да работам на мојата докторска дисертација и ги дискутиравме проблемите од мојот интерес во математичката статистика, негова сугестија беше да се обидам да применам алгебарски структури во теоријата на примерок верувајќи дека на тој начин многу од прашањата од интерес ќе можат поедноставно да се одговорат.

Се чувствувам среќна и благодарна што бев студент и соработник на професорот Чупона, што беше мој учител, академски и научен ментор и извонреден пријател.

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Review

RECOGNIZABLE AND REGULAR SUBSETS OF MONOIDS

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To the memory of Professor Gjorgji Čupona, with gratitude

This note is a short review of regular and recognizable subsets of monoids. We introduce a new question about characterizing classes of monoids and show that idempotent monoids can be characterized by the properties of the languages they recognize.

Key words: monoids, automata, regular languages, recognizable languages

INTRODUCTION

It is well known that there is an intimate relationship between regular languages (as subsets of free monoids) and their syntactic monoids (as transition monoids of the corresponding minimal deterministic automata). Many classes of languages can be characterized by the ideal structure of their corresponding syntactic monoids [17]. In particular, classes of regular languages that are studied in symbolic dynamics and cellular automata can be characterized through their transition monoids of the minimal deterministic presentation [9, 11]. With this note, we ask the converse question: can properties of the languages recognized by classes of finite monoids describe, or characterize, the class of monoids? For the simple case of idempotent monoids we show that such characterization is possible.

The notions of automata and languages can be extended to arbitrary monoids. One can consider M-regular subsets of M where M is an arbitrary monoid, not necessarily the free monoid. Similarly, the recognizable languages can be extended to M-recognizable subsets. In this case, the M-regular subsets may strictly contain the M-recognizable subsets and we show why this inclusion is strict. We end by recalling the long standing open problem for characterizing the monoids for which M-recognizable and M-regular sets coincide.

PRELIMINARIES

2.1. **Automata.** A standard background in automata theory can be found in [8, 20]. A monoid with identity 1 is denoted with M. A subset of a monoid M is called an M-*language*. The set of all words over a finite alphabet A is denoted by A^*. With the operation concatenation A^* is the free monoid generated by A. A *language* is an A^*-language.

Definition 2.1 Let M be a monoid. An *M-finite state automaton* (or just *M-automaton*) is a tuple $\mathcal{M} = (M, Q, I, T, \mathcal{E})$ where Q is a finite set of *states*, $I \subseteq Q$ the set of *initial* states, $T \subseteq Q$ the set of *terminal* states and $\mathcal{E} \subseteq Q \times M \times Q$ the set of *transitions*.

An M-automaton $\mathcal{M}$ is associated with a finite labeled directed multigraph having vertices Q, directed edges $\mathcal{E}$, and three functions, $s, t : \mathcal{E} \to Q$ (*source* and *target* of

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the edges) and the labeling $\lambda : \mathcal{E} \to M$ defined by $s(q,a,q') = q$, $t(q,a,q') = q'$ and $(\lambda(q,a,q') = a$. A *transition sequence* or a *path* in $\mathcal{M}$ is a sequence of edges

$$\begin{aligned} p = & \ e_1 e_2 \cdots e_k \\ = & \ (q_0,a_1,q_1)(q_1,a_2,q_2)\cdots(q_{k-1},a_k,q_k) \end{aligned}$$

satisfying $s(e_{i+1}) = t(e_i)$ for $i = 1,\ldots,k-1$. In fact $p \in \mathcal{E}^*$. The *label* of p is $\lambda(p) = \lambda(e_1)\cdots\lambda(e_k) = a_1 \cdots a_k \in M$. The *source* of p is $s(p) = s(e_1) = q_0$ and *target* of p is $t(p) = t(e_k) = q_k$.

A element $w \in M$ is *accepted* by $\mathcal{M}$ if there is a path p such that $s(p) \in I$, $t(p) \in T$ and $\lambda(p) = w$. In the case of $M = A^*$, w is called a word. The M-language *recognized by* $\mathcal{M}$ is $L(\mathcal{M}) = \{w \in M \mid w \text{ is accepted by } \mathcal{M}\}$. In particular, $1 \in L(\mathcal{M})$ if and only if $I \cap T \neq \emptyset$.

Definition 2.2 An M-language $L \subseteq M$ is *M-regular* if there exists an M-automaton $\mathcal{M}$ such that $L = L(\mathcal{M})$.

The class of M-regular languages is denoted $\mathcal{R}eg(M)$. Here we concentrate on *deterministic* M-automata, that is, for every $q \in Q$ and every $a \in M$ the set $\{q' \mid (q,a,q') \in \mathcal{E}\}$ is either a singleton or empty. It is well known that the class $\mathcal{R}eg(M)$ remains unchanged when we restrict our attention to deterministic automata. In the deterministic case, if $X \subseteq M$ is the set of labels of the transitions, then X^* (as a submonoid of M) acts on Q by $q \cdot a = q'$ or just $qa = q'$ for $(q,a,q') \in \mathcal{E}$ and $a \in X$. If there is no transition starting at q with label a then $qa = \emptyset$. One can always add a 'junk' state in Q and set $qa = junk$ whenever $qa = \emptyset$, hence, for each $a \in X$, its action on Q is considered as a function rather than a partial function. For $w \in X^*$, $qw = q'$ if there is a path in $\mathcal{M}$ from q to q' with label w. We usually take that $X^* = M$, i.e., X generates M. The *transition monoid* $\mathcal{T}(M)$ *of* $\mathcal{M}$ is the set X^* as functions acting on the states of $\mathcal{M}$.

2.2. **Monoids.** Let $L \subseteq M$ and $x \in M$. The *context of x in M with respect to L* is

$$C_L(x) = \{(u,v) \mid u, v, \in M, uxv \in L\}$$

We set $x \sim_L y$ if and only if $C_L(x) = C_L(y)$. The *syntactic semigroup* of L is the quotient $M/\sim_L$ denoted with $S(L)$ with the operation $[x][y] = [xy]$. A subset L of a monoid M is said to be *recognizable* if there is a morphism φ from M to a finite monoid N such that $L = \varphi^{-1}(P)$ for some subset $P \subseteq N$. A monoid N *recognizes* L if there is a morphism $\varphi : M \to N$ and a subset P of N such that $L = \varphi^{-1}(P)$. So L is recognizable if it is recognized by a finite monoid. The class of recognizable subsets of M is denoted by $\mathcal{R}ec(M)$. The following hold [17, 19].

Proposition 2.1

- *The syntactic monoid $S(L)$ recognizes L.*
- *If $L \in \mathcal{R}eg(A^*)$ then it is recognized by the transition monoid $\mathcal{T}(\mathcal{M})$ of the minimal deterministic $\mathcal{M}$. Moreover, $L \in \mathcal{R}eg(A^*)$ if and only if $L \in \mathcal{R}ec(A^*)$.*
- *A monoid M recognizes $L \subseteq A^*$ if and only if the syntactic monoid $S(L)$ of L divides M (it is a quotient of a submonoid of M).*

The ideal structure of a monoid can be described with the following equivalence relations which are based on the principal ideals.

Definition 2.3 Let M be a monoid. Green's relations R, L, J, H, D on M are defined as $(a, b \in M)$:

- $a\,R\,b$ if $aM = bM$
- $a\,L\,b$ if $Ma = Mb$
- $a\,J\,b$ if $MaM = MbM$
- $a\,H\,b$ if $a\,R\,b$ and $a\,L\,b$
- $a\,D\,b$ if there is $c \in M$ such that $a\,R\,c$ and $c\,L\,b$

In a finite monoid $D = J$. In this case the subgroups of M are the H classes containing idempotents.

MONOIDS AND LANGUAGES

3.1. **Monoid characterizations of classes of languages.** Algebraic characterization of languages is often used in automata theory, and some classes of languages show up in other fields, such as symbolic dynamics [14]. The concept of local languages remains fundamental in automata theory as every regular language is a morphic image of a local language (more precisely, strictly locally testable), and this characterization has been used to define regular 2-dimensional languages (sets of rectangular arrays of symbols [7, 12]). A language $L \subseteq A^*$ is *local* if it is a complement of finitely generated submonoid of A^*. In other words, L is local if there is a finite set of words F such that $L = A^* \setminus A^*FA^*$. The set F is called the set of *forbidden words*. A word $w \in A^*$ is called a *constant* for the language $L \subseteq A^*$ when for all $v_1, v_2, v_3, v_4 \in A^*$ the following implication holds,

$$v_1wv_2 \in L \text{ and } v_3wv_4 \in L \Rightarrow v_1wv_4 \in L.$$

A well known characterization of local languages states: k is the maximal length of the set of forbidden words if and only if all words of length $\geq k$ are constants for L [14, 11]. These words also act as constant functions (hence the name) in the action of A^* on the minimal deterministic automaton $\mathcal{M}$ recognizing the language.

Definition 3.1 A language $\emptyset \neq L \subseteq A^*$ is:

- *factorial* if for all $x, y, z \in A^*$
  $xyz \in L \Rightarrow y \in L$
- *extendable* if for all $x \in L$
  there are $y, z \in A^+$ such that $yxz \in L$
- *transitive* for all $x, y \in L$
  there is $z \in A^*$ such that $xzy \in L$

Local, factorial and extendable languages correspond to factors of subshifts of finite type, while factorial, extendable and regular (FER) languages consist of factors of sofic subshifts. Transitive languages can be associated with transitive symbolic dynamical systems [14]. These languages, in particular factorial, transitive and regular (FTR) languages can also be studied as factors of images and traces of cellular automata [2]. Their syntactic monoids can be characterized as follows. We set $\eta : A^* \to S(L)$ as the natural onto morphism defined with $x \mapsto [x]$.

Proposition 3.1 [9] *L is an FTR language if and only if $S(L)$ has the following properties:*

(i) $S(L)$ is finite

(ii) $S(L)$ has a 0 such that $\eta^{-1}(S(L) - \{0\}) = L$

(iii) $S(L)$ has a 0-minimal right ideal R (an R-class) such that for every non zero $x \in S(L)$, $Rx \neq 0$.

In this case, one can define an A^*-automaton such that the states are the R-classes of the 0-minimal right ideal of $S(L)$ with transitions defined as $[x]a = [xa]$. This automaton, in fact, becomes the minimal transitive representation of the language [9]. Let $\mathcal{I}_c = \{[x] \,|\, x$ is a constant for $L\}$. Observe that $\mathcal{I}_c$ is an ideal for $S(L)$. Moreover, for a local, or an FTR-language L, the word $c \in L$ is constant if and only if $D_{[c]}$ has H-trivial subclasses, that is, the corresponding D-classes are group-free. This holds even for a larger class of languages that are factorial and extendable (not necessarily regular), such as the Dyck languages. Although their syntactic monoids are infinite, the classes of relations D and J coincide [10].

Proposition 3.2 [11, 15] *Let L be a language and $\mathcal{I}_c = \{[x] \,|\, x$ is a constant for $L\}$. The language L is local if and only if $[1] = \{1\}$ and the set of idempotents $E = \{e \,|\, e^2 = e, e \neq 1, e \neq 0\}$ is a non-empty subset of the ideal $\mathcal{I}_c$.*

Note that every finite group can be a syntactic monoid of some language by an appropriate definition of an action of the group to a directed graph.

3.2. Language characterization of classes of monoids.

As computer science, and in particular, algebraic automata theory concentrates on understanding classes of languages, there has been virtually no studies of the converse question. *Characterize classes of monoids according to the classes of languages that they recognize.* In particular, consider the class $\mathcal{C}$ of finite monoids that belong to an identity defined variety of monoids, such as the variety defined by $x^n y^n = (xy)^n$ (varieties of such groupoids and other algebraic structures have been studied by Čupona and his collaborators, e.g. [3, 4]). Can the properties of the classes of languages that are recognized by $\mathcal{C}$ determine the monoids in $\mathcal{C}$? One simple case with a positive answer can be observed with the following. Recall that idempotent monoids are monoids whose every element is an idempotent.

Proposition 3.3 *A finite monoid M is an idempotent monoid if and only if every language L recognized by M satisfies the equivalence*

$$w \in L \Longleftrightarrow w^+ \subseteq L$$

Proof. If M is an idempotent monoid then M satisfies the equation $x^2 = x$. Let $\eta : A^* \to M$ be a morphism and L recognized through $P \subseteq M$ such that $L = \eta^{-1}(P)$. Then for $w \in L$, and $\eta(w) = p \in P$ we have that $pp = \eta(ww) = \eta(w^n) = p$, hence, for all n, $w^n \in \eta^{-1}(p)$, which implies that $w^+ \subseteq L$. Of course, if $w^+ \subseteq L$ then $w \in L$ by definition.

Converse, suppose that every L recognized by M satisfies the equivalence of the proposition. Consider a surjective morphism $\eta : A^* \to M$. Let $p \in M$ and take P to be the singleton $P = \{p\}$ and let $L = \eta^{-1}(p)$. Suppose w is of minimal length such that $\eta(w) = p$. The equivalence of the proposition says that $w^+ \subseteq L$ and hence $\eta(w^n) \in P$, i.e. $\eta(w^n) = p$. In particular, for $n = 2$ we have $\eta(ww) = pp = p$, implying that p must be an idempotent. As this is true for every $p \in M$, M is an idempotent monoid. $\square$

For a finite idempotent monoid and $P \subseteq M$, let $\eta^{-1}(p)_{\min}$ be the minimal length $w \in A^*$ such that $\eta(w) = p$. Then whenever we have $p_1, p_2, p_1 p_2 \in P$ the language recognized as $\eta^{\ 1}(P) \subseteq A^*$ contains the subsemigroup $(\eta^{-1}(p_1)_{\min} \cup \eta^{-1}(p_2)_{\min})^+$ which is extendable and transitive.

As a specific case, consider the idempotent monoid $\mathcal{J}_3 = \{1, h_1, h_2, h_1h_2, h_2h_1\}$ satisfying $h_i h_j h_i = h_i$ for $i, j = 1, 2$. It consists of a single D-class containing the four non-identity elements, two R-classes ($\{h_1, h_1h_2\}$, $\{h_2h_1, h_2\}$) and similarly two L-classes. If $A = \{a, b\}$ the only languages recognized by $\mathcal{J}_3$ are a^ϵ, b^ϵ, aA^ϵ, bA^ϵ, $A^\epsilon a$, $A^\epsilon b$ where $\epsilon \in \{*, +\}$ and their pairwise intersections and unions.

We point out that the above question (characterizing classes of monoids through the properties of the languages they recognize) can be considered also for classes that are not necessarily varieties. The monoid $\mathcal{J}_3$ is a special case of the Jones monoids $\mathcal{J}_n$ generated by h_i, $i = 1, \dots, n-1$, with relations (A) $h_i h_j h_i = h_i$ for $|i-j| = 1$, (B) $h_i h_i = h_i$ and (C) $h_i h_j = h_j h_i$ for $|i \quad j| \geq 2$. Unfortunately, except for $n = 3$, the Jones monoids are not idempotent monoids, but they are all finite monoids [1, 21]. Given the relations (A)–(C), what are the properties of the languages recognized by $\mathcal{J}_n$? Can those properties be listed such that monoids $\mathcal{J}_n$ can be characterized?

RECOGNIZABLE VS REGULAR

Kleene's theorem says that $\mathcal{R}eg(A^*)$ coincides with the smallest class of languages that contain all finite languages and is closed under union, product and $*$-operation ($L^* = \cup_{i=0}^{\infty} L^i$ where $L^0 = \{1\}$) [13]. Similarly, the smallest class of subsets of M that contains all finite subsets of M, is closed under union, product and $*$-operation is the set of rational M-languages denoted $\mathcal{R}at(M)$. It can be observed that $\mathcal{R}at(M) = \mathcal{R}eg(M)$ for every M [5].

Consider $A = \{a, b\}$ and $M = A^* \times A^*$. A two state automaton (states q_1, q_2 with transitions $q_1(a, 1) = q_2$ and $q_2(1, b) = q_1$, having initial and terminal state q_1) recognizes the M-language $L_M = (a, b)^* = \{(a^n, b^n) \mid n \geq 0\}$. One can add a junk state q_3 sending all other missing transitions with generators $(a, 1), (b, 1), (1, a), (1, b)$ to this state. Hence $(a, b)^* \in \mathcal{R}eg(M)$. Let N be a finite monoid and a morphism $\eta : M \to N$. Let $\eta(a, 1) = x$ and $\eta(1, b) = y$. Because $(a, 1)(1, b) = (1, b)(a, 1) = (a, b)$ we must have $xy = yx$ in N. If $P \subseteq N$ is such that $\eta^{-1}(P) = L_M$, then P must contain the submonoid of N generated by xy. However, due to the commutativity, $(xy)^n = x^n y^n$. Because N is finite, there are n, k such that $x^{n+k} = x^n$ and therefore $\eta(a^{n+k}, b^n) = x^{n+k} y^n = x^n y^n \in P$. But $(a^{n+k}, b^n) \notin L_M$. Thus $L_M \notin \mathcal{R}ec(M)$. Observe that *there is no morphism* from M to the transition monoid of this M-automaton since the action of $(a, 1)(1, b)$ on the states q_1, q_2, q_3 is not the same as the action of $(1, b)(a, 1)$, although they represent the same element in M.

By Proposition 2.1 we have that $\mathcal{R}eg(A^*) = \mathcal{R}ec(A^*)$. The equivalence of two automata, or the emptiness problem for A^*-automata are easily decidable. $\mathcal{R}ec(A^*)$ is closed under intersection, but in general $\mathcal{R}ec(M)$ is not necessarily so, and that poses the main problem in understanding $\mathcal{R}ec(M)$. The simple example above is a base of several undecidability observations.

Proposition 4.1

(a) There is a monoid M such that it is undecidable whether $L_1 \cap L_2 = \emptyset$ for $L_1, L_2 \in \mathcal{R}eg(M)$ [18].

(b) There is a monoid M such that it is undecidable whether two M-automata recognize the same language [6].

In general, for a finitely generated monoid M, $\mathcal{R}ec(M) \subseteq \mathcal{R}eg(M)$ [16], but, as we saw above, the other inclusion does not necessarily hold. If M is not finitely generated, then even this inclusion does not hold. Take $M = \mathbb{Z}$ with the multiplication of numbers as the operation. One can map $\mathbb{Z}$ into a three element monoid $\{0, 1, 2\}$ with 0 being a zero element, 1 being an identity, and $2 \cdot 2 = 0$. The map sends $1, -1 \mapsto 1$, prime $\mapsto 2$, and non-prime $\mapsto 0$. The inverse image of $\{2\}$ is the set of primes (hence the set is recognizable), but it cannot be a regular set since no automaton can recognize this set.

When M is a group we have the following:

Proposition 4.2. [19] *Let G be a finitely generated group and $H \leq G$. Then*

(a) $H \in \mathcal{R}eg(G)$ iff H is finitely generated

(b) $H \in \mathcal{R}ec(G)$ iff H has finite index.

Problem: Characterize Kleene's monoids, i.e., monoids with $\mathcal{R}ec(M) = \mathcal{R}eg(M)$.

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REFERENCES

[1] M. Borisavljević, K. Došen, Z. Petric, Kauffman monoids, *Journal of Knot Theory and its Ramifications*, **11** 2 (2002), pp. 127–143.

[2] J. Cervelle, E. Formenti, P. Guillon, *Sofic Trace Subshift of a Cellular Automaton*, (B.Löwe S.B.Cooper and A.Sorbi, eds.) Computation and Logic in the Real World, CiE'07 Springer LNCS, **4497** (2007), pp. 152–161.

[3] Gj. Čupona, N. Celakoski, Free groupoids with $(xy)^2 = x^2y^2$, *Contributions, Sec. Math. Tech. Sci.,*MASA, **XVII**, 1-2 (1996), pp. 5—17.

[4] Gj. Čupona, N. Celakoski, S. Ilić, On monoassociative groupoids, *Mat. Bilten*, **(26)** (2002), pp. 5—16.

[5] C. C. Elgot, J. E. Mezei, On relations defined by generalized finite automata, *IBM J. Res. and Develop.,* **9** 36, 44 (1965), pp. 47–68.

[6] P. C. Fischer, A. L. Rosenberg, Multitape one-way nonwriting automata, *J. Computer System Sci.*, **2** 37 (1968), pp. 88–101.

[7] D. Giammarresi, A. Restivo, Recognizable picture languages, (M. Nivat, A. Saoudi, P.S.P. Wang, eds.) *Proc. 1st Internat. Colloq. on Parallel Image Processing, Internat. J. Pattern Recognition Artif. Intell.*, **6** (1992), pp. 231–256.

[8] J. E. Hopcroft, J. Ullman, *Introduction to Automata Theory, Languages, and Computation*, Addison-Wesley, 1979.

[9] N.Jonoska, Sofic Shifts with Synchronizing Presentations, *Theoretical Computer Science*, **158** 1-2 (1996), pp. 81–115.

[10] N. Jonoska, Constants in factorial and prolongable languages, *Pure Math. Appl.*, **7** 1-2 (1996), pp. 99–110.

[11] N. Jonoska, A conjugacy invariant for reducible sofic shifts and its semigroup characterizations, *Israel Journal of Mathematics*, **106** (1998), pp. 221–249.

[12] N. Jonoska, J.B. Pirnot, Transitivity in Two-Dimensional Local Languages Defined by Dot Systems, *International Journal of Foundations of Computer Science*, **17** 2 (2006), pp. 435–464.

[13] S. C. Kleene, Representation of events in nerve nets and finite automata, (C. Shannon, J. McCarthy, eds.) *Automata Studies*, Princeton Univ. Press, **2** (1956), pp. 3–41.

[14] D. Lind, B. Marcus, *An Introduction to Symbolic Dynamics*, Cambridge University Press, New York, 1995.

[15] A. de Luca, A. Restivo, A characterization of strictly locally testable languages and its applications to subsemigroups of a free semigroup, *Information and Control*, **44** (1980), pp. 300–319.

[16] J. McKnight, Kleene's quotient theorems, *Pacific J. Math.*, **36** 14 (1964), pp. 43–52.

[17] J-E. Pin, Syntactic semigroups, (G. Rozenberg, A. Salomaa, eds.) *Handbook of language theory,*, Springer Verlag, Vol. **I**, (1997), pp. 679–746.

[18] M. O. Rabin, D. Scott, Finite automata and their decision problems, *I.B.M. J. Res. Develop.*, **3** (1959), pp. 125–144. Reprinted in *Sequential Machines : Selected Papers* (E. Moore, ed.), Addison-Wesley, 1965

[19] J. Sakarovitch, *Elements of Automata Theory*, Cambridge University Press, 2009.

[20] A. Salomaa, *Lewels of Formal Languages*, Computer Science Press, 1981.

[21] K. Wai Lau, D. G. FitzGerald, Ideal Structure of the Kauffman and related monoids, *Communications in Algebra*, **34** 7 (2006), pp. 2617–2629.

ПРЕПОЗНАТЛИВИ И РЕГУЛАРНИ ПОДМНОЖЕСТВА ОД МОНОИДИ

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Во спомен на професор Ѓорѓи Чупона со длабока благодарност

Овој запис е кратка ревизија на регуларни и препознатливи подмножества од моноиди. Воведуваме ново прашање за карактеризација на класи моноиди и покажуваме дека класата на иденпотентни моноиди може да се карактеризира преку својствата на јазиците препознаени со оваа класа.

Клучни зборови: моноиди, автомати, регуларни јазици, препознатливи јазици

Имав привилегија професор Чупона да ми помогне да го започнам моето професионално патешествие и му должам голема благодарност. Мојата дипломска работа беше на тема комбинаторна теорија на групи и професор Чупона беше мој ментор. Подоцна за време на моите докторски студии се навратив на оваа тема, а и мојата докторска дисертација заврши со добар дел посветен на полугрупи. Како асистент на Институтот по Математика за две-три години, професор Чупона ми ја препорача книгата Теорија на Автомати од Арто Салома [20], и таа област заврши како главна тема на мојата дисертација и истражување ([9, 10, 11]). А повеќе од сè беа моментите на дружење што тој ги креираше и што не правеа блиски со него, а и блиски меѓу нас како другари математичари.

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Educational paper

THE ROLE OF VISUALIZATION IN UNDERGRADUATE MATHEMATICS

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We emphasize the importance of visualization in undergraduate mathematics courses and suggest drawing-to-learn intervention that will help students solidify concept images of mathematical objects through drawing activity.

Key words: visualization, visualization object, undergraduate mathematics

INTRODUCTION

Visualization as a process of reflection upon pictures, images or diagrams on a blackboard, paper or other technical tools, is a subject of intense research in the last decade. Deductive and analytical nature of mathematical reasoning seems detached from the influence of visual images designed to illustrate the connection between the given data and the unknown in a particular mathematical problem. In recent years we are witnessing a tremendous development of computer graphics and educational software. Their ubiquitous penetration into educational practice may lead some to believe that nonparticipative exposure of the learner to predetermined images and spatial representations of mathematical objects on a computer screen can be an alternative to the process of active creation of visual representation of mathematical objects by hand, and opportunity to manipulate them in this creative process. The process of sketching abstract mathematical objects involves powerful hand-mind coordination which results in concretization of these objects as creations of our own hands, subject to easy manipulation or transformation. Although we are witnessing an increasing demand for profiles in science, technology, engineering and mathematics (STEM), more than 40% of STEM majors in US universities switch to non-STEM majors before graduation [8]. Mathematics education researchers' attempt to analyse this problem, among other factors, emphasize the weakening standards of high school mathematical curriculum and not enough attention has been paid to different modes of presenting mathematical content.

Inspiration for this essay is the work of Quillin and Thomas [15] in which they create a framework for drawing-to-learn approach to reasoning in biology classroom. Our research in using visualisation in teaching of Calculus III, Vector Calculus and Linear Algebra courses taught at university level shares many of their findings and suggestions. We have demonstrated that the reluctance to use visualization as a tool in problem-solving strategies is not correlated with students' ability to sketch, but rather to the predominance of the analytic way of presenting mathematical statements in the school curriculum. Hesitancy towards adequate use of visual arguments in the process of justification (proof), adds to the perceived bias towards visualizing mathematical statements, and deprives students of a powerful cognitive tool. Freehand drawing of mathematical objects (lines, planes, spheres, etc.) has all the elements of modelling and creates an opportunity for a learner to manipulate an abstract mathematical object and serves as a cognitive tool in the learning process.

WHAT IS VISUALIZATION

There is no unified definition of the term "visualization" in mathematics education literature. In their influential work on visualization in mathematics, reading and science education Phillips, Norris and Macnab give 28 explicit definitions of visualisation and related terms, in education literature from 1974 to 2010 [12]. In the educational literature one can find multiple usages for the same term, sometimes even contradicting each other. In what follows, we adapt their definition of "visualization object", but we use the term "visualization" in drawing-to-learn activity as a verb.

Visualization objects are physical objects that are viewed and interpreted by a person for the purpose of understanding something other than the object itself. These objects can be pictures, 3D representations, schematic representations, animations, etc. Other sensory data such as sound can be integral parts of these objects and the objects may appear on many media such as paper, computer screens and slides.

Bishop [2] was the first one to note the important distinction between use of the term "visualization" as a noun and as a verb. The noun "directs our attention to the product, the object, the 'what' of visualization, the visual images. The verb of visualization, on the other hand, make us attend to the process, the activity, the skill, the 'how' of visualizing". He defines "visual processing ability" as "ability that involves visualization and the translation of abstract relationships and non-figural information into visual terms. It also includes manipulation and transformation of visual representations and visual imagery. It is an ability of process and does not relate to the form of the stimulus material presented." [2]

Advancement of electronic devices and tools for drawing and computer-generated animations necessitated modification of this definition, resulting in the above-mentioned definition in [12].

Like in the case of visualization, there is no unified approach or definition of drawing in drawing-to-learn notion. One can adopt an inclusive definition of drawing given in [15] for the purposes of drawing mathematical objects or mathematical notions, broadly defined as:

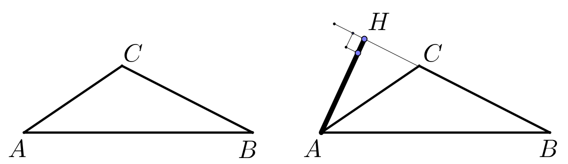
a learner-generated external visual representation depicting any type of mathematical object, whether structure, relationship, or process, created in static two dimensions on any medium.

We should note that creating an external model of a mathematical object requires not only mental processes, but also coordination of hand movements created by following some kinematic and kinetic parameters that will result in intended action. The point of mentioning this is that drawings presented by an experienced instructor can be intimidating for an unexperienced learner, thereby students should be constantly encouraged and reminded that artistic attributes of the visualized object in most cases are not a prerequisite for its successful use in the cognitive process.

Although in our teaching practice we are mostly focused on constructing visualization objects, either on a whiteboard during enacted lesson, on paper, or electronically on a computer screen, we should mention two other distinctive attributes of visualization as a process.

Introspective visualization refers to mental objects that a person/learner makes in building their *concept image*. The notion of a concept image and *concept definition* are two useful ways of understanding a mathematical concept. These were created by Tall and Vinner [18] and often visualization is discussed in the framework suggested in [18]. They define the notion of a concept image "…to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures". A concept definition on the other side, is similar to the notion of definition in mathematics, with the distinction of being personal to an individual. According to [18], "… a personal concept definition can differ from a formal concept definition, the latter being a concept definition which is accepted by the mathematical community at large".

Interpretive Visualization is an act of making meaning from a visualization object or an introspective visualization by interpreting information from the objects or introspections and by cognitively placing the interpretation within the person's existing network of beliefs, experiences, and understanding [12]. Our pedagogical practice shows that many high school and collegiate geometry students do not make the distinction between a mathematical object (notion) and their physical realization in the form of a visualization object or picture. Just as an illustration, if *AH* is the altitude from vertex *A* in the triangle *ABC* where we assume that the angle at the vertex *C* is an obtuse angle, for majority of geometry students, the altitude *AH* will not exist as a mathematical object, or it will not be introspectively visualized, unless drawn on the paper or a whiteboard.



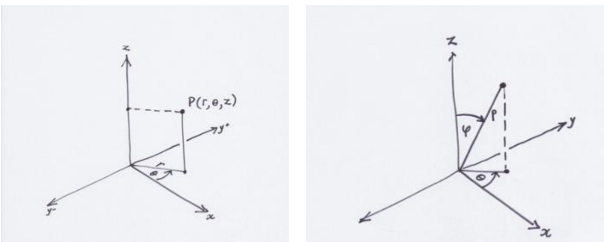
Introducing the altitude *AH* as an auxiliary element in the visual representation of the triangle *ABC*, will provide valuable insight on how to apply the basic formula for the area of a triangle if we take side *BC* to be a base of the triangle [13] (see p. 47).

WHAT CONSTITUTES A GOOD VISUAL INTERPRETATION?

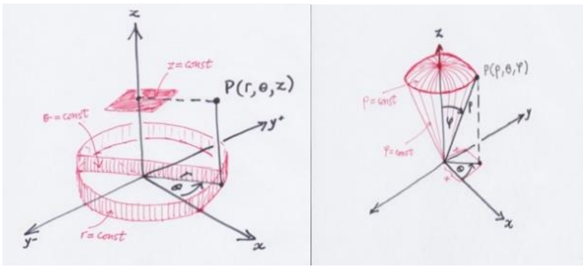
Research has demonstrated that for the vast majority of scientists and mathematicians' visualization plays central role in their cognitive processes. Comprehensive theory of visual images in mathematical education is still lacking, and in absence of such theory, we rely more on intuition and scattered evidence on the use of images in the learning process or usefulness of drawing-to-learn activities embodied in the classroom practice.

In what follows, we will outline number of interventions that will help instructors create an environment conductive to students' drawing-to-learn activities in the classroom. Our experience and research have been conducted with students in Calculus and Linear Algebra classes. On few occasions we've worked with College of Education's prospective teachers in the Geometry Connection course. More information about research methods and findings can be found in [3,5,10,11].

Typical Calculus III material is especially suitable for drawing-to-learn approach. The example that follows was from enacted lesson to 45 engineering students and more details can be found in [10]. We note that students had previous experience with drawing 3D coordinate system and sets of points whose coordinates satisfy certain (simple, mostly linear) algebraic equation(s). We distinguished three categories of images presented during enacted lesson: primary image, secondary image, and second-layer image. We define *primary image* as an image on which the derivation of the analytical portion of the presentation rests, also related to a justification (visual proof) of subsequent proposition. In the following figure we provide two examples of primary images that were given in the lecture about triple integrals in spherical coordinate system.



Primary image may also play an essential role in the explanation of a new mathematical concept. Usually, this image will stay on the board during an enacted lesson for a substantial amount of time (compared with the length of the class period). A *second-layer image* is an image that will be superimposed on a primary image later in the exposition, bringing new aspects of the presented notion, or illustrating a portion of the proof of a proposition. Most of the time, during the enacted lesson, students are inclined to sketch a completely new illustration rather than revisiting a primary image and superimposing on it a new one.

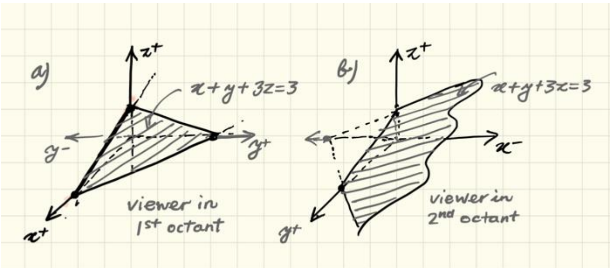


Illustrative elements that have been superimposed on the primary images are indicated as shaded areas in red. Two remarks are in place for the second-layer images. The first one is that we follow Inglis and Mejia-Ramos' suggestion that students are more inclined to accept figures as evidence for a claim if it is accompanied by descriptive text explaining the claim. In the enacted lesson, the second layer imagery has been used also to reinforce the previous notions (cylindrical coordinate system) and emphasize connections between the coordinate systems. Students were invited to derive analytical/algebraic expressions for (portions of the) surfaces shown and convince themselves about justification of introducing these new coordinate systems. The second note is about the use of color when presenting images of mathematical objects. Research shows that excessive use of color can be impeding to the cognitive process and could be experienced as a distraction to the learner.

Our third category of images are so called *secondary images*. These are images that have been used to clarify a particular argument related to the primary image, illustrate a particular point in the analytical portion of the argument, or used as a review of a specific notion used in the exposition. Usually, these images will stay a short time on the board, serving its purpose and not interfering with the primary image. We illustrate this category in the following figure.

The left picture on the above figure helps students recall the definition of sine and cosine function but in an unusual way. Usually, these functions are given as ratios of a length of a leg with length of the hypothenuse in a right triangle. Our practice shows unusual persistence of this high school concept image and prevents students to see different forms of this analytical formula. The right picture illustrates (parts of) four graphs of the equation $\varphi = c$ (c is a constant) in the spherical coordinate system, for four different choices of c all between 0 and π. Initially, after showing the cone with c close to zero, students are invited to sketch a cone with their specific choice of c.

One can notice that on previous figures, mathematical objects are presented in their "typical" position. In [5] we have examined the diversity of imagery of the same mathematical object (triangle, parallelogram and trapezoid) in high school geometry books used in majority of schools in Florida. Our definition of a typical images of a particular mathematical object is "…*a visual representation of that object that is drown in the majority of instances with no content-based reason"*. In our view on the 3D coordinate system, the viewer is in the first octant and this is the typical representation in calculus textbooks. The role of the instructor is to point this anomaly and to invite students to represent the same mathematical object in 3D but from different standpoint of the observer.



We have illustrated the need to draw in calculus classes, especially when working with undergraduate STEM majors, but similar arguments can be made that will advocate the use of visual arguments in mathematics classes, for much needed scaffolding when constructing proofs of a given propositions. To be clear, we are not advocating acceptance of visual argument and pictures, as proofs in mathematics. We seek interventions that will help student in establishing well organized and coherent library of concept images, as a necessary tool in the practice of proving mathematical theorems.

REFERENCES

[1] A. Arcavi, The role of visual representations in learning of mathematics, *Educational Studies in Mathematics*, **52** (3) (2003), pp. 215–241.

[2] A.J. Bishop, Review of research on visualization in mathematics education, *Focus on Learning Problems in Mathematics*, **11** (1) (1989), pp. 7–16.

[3] D.K. Birinci, M. Krajcevski, G. McColm, Assessing Visual Literacy Competency in Undergraduate Mathematics, *21st Annual Conference on Research in Undergraduate Mathematics Education*, San Diego, CA (1273-1278), 2018.

[4] G. Brousseau, *Theory of didactical situations in mathematics*, Kluwer Academic Publishers, 1997.

[5] M. Cannon, M. Krajcevski, On prevalence of images in high school geometry textbooks, *Proceedings of the Second International Conference on Mathematics, Textbooks Research, and Development, 2017 Rio de Janeiro*, (2018), pp. 180–190.

[6] R. Duval, A cognitive analysis of problems of comprehension in learning of mathematics, *Educational Studies in Mathematics*, **61**(1) (2006), pp. 103 131.

[7] M. Inglis, J.P. Mejia-Ramos, On the persuasiveness of visual arguments in mathematics, *Foundations of Science*, **14** (1-2) (2009), pp. 43 110.

[8] K. Koenig, M. Schen, M. Edwards, and L. Bao, Addressing STEM Retention Through a Scientific Thought and Methods Course, *Journal of College Science Teaching*, **41**(4) (2012), pp. 23–29.

[9] M. Krajcevski, Using drawing as a cognitive strategy in undergraduate education, (Unpublished) (2016)

[10] M. Krajcevski, K. Keene, Students' inclination to use visual images during problem solving, *20th Annual Conference on Research in Undergraduate Mathematics Education*, San Diego, CA (1273-1278), 2017.

[11] M. Krajcevski, R. Sears, Common Visual Representations as a Source for Misconceptions of Preservice Teachers in a Geometry Connection Course, *International Journal for Mathematics Teaching and Learning*, Vol **20.1** (2019), pp. 85–105.

[12] L.M. Phillips, S.P. Norris, J.S. Macnab *Visualization in Mathematics, Reading and Science Education*, Models and Modeling in Science Education, Springer, 2010.

[13] G. Polya, *How to Solve It (2nd Ed.)* Princeton University Press, Princeton, NJ, 1957.

[14] N.C. Presmeg, Research on visualization in learning and teaching mathematics, in: *Handbook of research on psychology of mathematics education*, A. Gutieras and P. Boero (Eds), 2006, pp. 205–235.

[15] K. Quillin, S. Thomas, Drawing-to-Learn: A Framework for Using Drawings to Promote Model-Based Reasoning in Biology (Essay) *CBE-life Sciences Education*, **14** (2015), pp. 1–16.

[16] D. Stylianou, D. Pitta, Representations and Mathematics Visualization, in: *North American Chapter of the International Group for the Psychology of Mathematics Education*, F. Hitt (Ed), 2002, pp. 31–47.

[17] D. Tall, M.P. Mejia-Ramos, The long-term cognitive development of reasoning and proof, in: *Explanation and Proof in Mathematics, Philosophical and Educational Perspectives*, G. Hanna, H. Jahnke, H. Pulte (Eds), Springer, 2010, pp. 137–149.

[18] D. Tall, S. Vinner, Concept image and concept definition in mathematics with particular reference to limits and continuity, *Educational Studies in Mathematics*, **12**(2) (1981), pp. 151–169.

[19] S.P.W. Wu, M.A.Rau, How students learn content in: *Science, Technology, and Mathematics (STEM) through drawing activities, Educational Psychology,* Review Published Online January 15, (2019)

УЛОГАТА НА ВИЗУАЛИЗАЦИЈАТА ВО НАСТАВАТА ПО МАТЕМАТИКА ВО ВИСОКОТО ОБРАЗОВАНИЕ

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Овој краток есеј ја обработува темата на визуализација на математичките објекти и поими во методиката на наставата по математика во високото образование. Покрај дефинициите на визуализација и поимот цртам-да-научам, низ конкретни примери се илустрира важноста на овој начин на презентација на математичка содржина особено во техничките науки.

Клучни зборови: визуализација, визуелен објект, високо образование

Има многу да се раскажува за влијанието на професор Чупона не само на мојот пат во математиката, туку и на патот на една цела генерација од македонски математичари и педагози. Тој имаше дарба да почувствува кои математички прашања би биле интересни за соговорникот и не ги наметнуваше неговите погледи како нешто што треба да се следи. Но пред се, имаше визија како треба да се унапредува и развива математичката мисла во Македонија и умешно им сугерираше на студентите каде нивниот потенцијал најдобро ќе дојде до израз. Беше свесен за потребата на дидактичките и методичките истражување во математиката и ми сугерираше да работам на проблемите поврзани со наставата по математика во високото образование.